Integral and fractional Quantum Hall Ising ferromagnets

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We compare quantum Hall systems at filling factors $\nu = 2$ to $\nu = \frac{2}{3}$ and $\frac{2}{5}$, corresponding to the exact filling of two lowest electron or composite fermion (CF) Landau levels. The two fractional states are examples of CF liquids with spin dynamics. There is a close analogy between the ferromagnetic (spin polarization $P = 1$) and paramagnetic ($P = 0$) incompressible ground states that occur in all three systems in the limits of large and small Zeeman spin splitting. However, the excitation spectra are different. At $\nu = 2$, we find spin domains at half-polarization ($P = \frac{1}{2}$), while antiferromagnetic order seems most favorable in the CF systems. The transition between $P = 0$ and 1, as seen when e.g. the magnetic field is tilted, is also studied by exact diagonalization in toroidal and spherical geometries. The essential role of an effective CF–CF interaction is discussed, and the experimentally observed incompresible half-polarized state is found in some models.

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I. INTRODUCTION

The long range spin order in quantum Hall systems at integer filling factor can easily be explained in terms of single electron Landau levels and it became common to call these systems quantum Hall ferromagnets[25] (QHF). Electron-electron repulsive interaction, in particular exchange, is known to further stabilize the ferromagnetism. Given the composite fermion (CF) mapping[7], it was not surprising that ground states (GS) with spin order were found also at fractional filling factors which correspond to integer filling factor of CFs. However, since the interactions between CFs are different to the previous case, it was not clear whether they promote or suppress the ferromagnetism. Moreover, the interaction can be more important in the fractional regime, and it might even destroy the ferromagnetism, especially near a transition between two GS of different order, because it cannot be downscaled as in the integer regime for $B \to \infty$. The field of fractional QHF became particularly interesting when signatures of states with intermediate polarization were experimentally discovered near the transition.

The mentioned GS transition in systems at filling factor $\nu = n/(eB/h) = 2$ occurs when the $0\uparrow$ and $1\downarrow$ Landau levels (LL) cross[8,38]. This happens when we vary the ratio of Zeeman and cyclotron energies (θ) as it is the case with tilting the magnetic field B, changing the g–factor or pumping the nuclear spins of the host lattice. The $0\downarrow$ LL is always full at filling factor two, while $1\downarrow$ and $0\uparrow$ is full and empty for large θ and vice versa for small θ. The GS is thus fully spin polarized in the first case and it is a spin singlet in the second case. The stabilizing effect of interactions implies that the transition between these two spin–ordered states is abrupt without any intermediate state when θ is varied.

The situation is different at filling factors $\nu = \frac{2}{5}$ and $\frac{2}{3}$ which both correspond to filling factor $\nu^* = 2$ of composite fermions (in the latter case, the effective magnetic field acting on CFs points in opposite direction to B). The crossing of $0\uparrow$ and $1\downarrow$ CF LLs is now induced by varying the ratio of Zeeman and Coulomb energy $\eta \propto \sqrt{B}$, since the CF cyclotron energy ($\hbar\omega_c^*$) is determined fully by the electron–electron interaction if LL mixing is neglected. Optical experiments by Kukushkin *et al.*[28] confirmed the transition from $P = 0$ to $P = 1$ when η was increased, but they also revealed a stable intermediate state at $P = 0.5$. Experiments by Freytag *et al.*[22] suggested another intermediate state with $P \approx 0.8$. On the other hand, transport measurements[24,26,27,35,36] showed huge longitudinal magnetoresistance at the transition, which was attributed to domain formation as an opposite to a homogeneous incompressible quantum Hall state.

Possible stable half–polarized states in the context of $\nu = \frac{2}{5}$ and $\frac{2}{3}$ were then discussed by Apal'kov *et al.*[20] (condensate of $L = 1$ excitons), Murthy[31] using the Hamiltonian theory[32] of CF (quantum Hall crystals), Mariani *et al.*[29] and Merlo *et al.*[30] (pairing of CFs similar to superconductivity) and more recently also by Yang *et al.*[39] (unidirectional CDW of CF). Spin transitions and instabilities were also studied in other QH systems including $\nu = 2$ (Giuliani *et al.*[8]), $\nu = 4/3$ (one of the authors[38]) or higher integer fillings (Rezayi *et al.*[34]).

The purpose of this article is primarily to compare systems of electrons and composite fermions at the same filling factor ν or ν^* equal to two. We numerically investigate the *electrons* at filling $\nu = 2$ versus $\nu = \frac{2}{5}$ and $\frac{2}{3}$. We show that even though the ground states are analogous, the excitations are quite different. As a consequence, physics of the paramagnet-ferromagnet transition is distinct in the two systems. A low-energy half-polarized state with antiferromagnetic spin order is found at $\nu = \frac{2}{3}$, while domains of $P = 0$ and $P = 1$ are found at $\nu = 2$. Next we turn to the concept of composite fermions (CF). It is demonstrated that the effective interaction between

the CFs calculated near $\nu = \frac{1}{3}$ (or $\nu^* = 1$) leads to very questionable results when applied to $\nu = \frac{2}{5}$, $\frac{2}{3}$ and we discuss alternative approaches.

A. Exact diagonalization

Each Landau level (LL) is highly degenerate, in a given area it can accommodate (maximum of) N_m electrons of exactly the same energy. When studying the quantum Hall ferromagnets (QHF), it happens frequently that N particles, electrons or composite fermions, are to occupy these states. In absence of interaction this yields a vast number of degenerate N–particle states: $\binom{N_m}{N}$. With particle–particle interaction switched on, no perturbation theory is tractable, as it requires a unique ground state to start with. The standard way to handle this problem is to diagonalize the full Hamiltonian with respect to the full basis of dimension $\binom{N_m}{N}$. The Hamiltonian comprises primarily of the particle–particle interaction and the fundamental approximation made is that we consider a finite system of N particles rather than an infinite one.

Within Haldane's model[1], N electrons are confined to a spherical surface of radius R. Dirac monopole of strength $2Q$ (in the units of elementary flux quantum, hc/e) in the center acts as a source of radial (i.e., normal to the surface) magnetic field B. Magnetic length scale $\ell_0 = \sqrt{hc/eB}$ is simply related to $2Q$ and R by $R^2 = Q\ell_0^2$.

The lowest (0th) Landau level (LL) is a shell of angular momentum $l_0 = Q$ and finite degeneracy $g_0 = 2l_0 + 1$ (with different orbitals distinguished by angular momentum projection m). Higher LL's, labeled by $n > 0$, have $l_n = Q + n$ and $g_n = 2l_n + 1$. Including spin, the single-particle states on Haldane sphere (called "monopole harmonics")[2] are uniquely denoted by $i = [n, m, \sigma]$. The dependence of LL degeneracy g_n on the LL index n is a finite-size artefact of spherical geometry, known to cause some inconvenience in calculations involving different LL's.

The integration of two-body interaction (Coulomb) matrix elements $\langle i, j|V|k, l\rangle$ can be done analytically for an ideally 2D system[3]. For finite width w of the electron layer, a fixed density profile $\varrho_w(z)$ can be used to model the lowest subband in the normal direction, and the calculation of $\langle i, j|V|k, l\rangle$ involves one-dimensional numerical integration. These two-body matrix elements are related with Haldane interaction pseudopotential (pair interaction energy as a function of relative angular momentum $\mathcal{R}$)[4] through the Clebsch-Gordan coefficients.

A complication with the spherical geometry is the definition of the filling factor: $\nu = N/(2l_0+\gamma)$. The "shift" γ is a topological quantum number[46] which is of the order of one, independent on N but it need not be the same for different states at the same value of ν. Thus, looking for one particular state at a given filling factor, we must also know its γ and adjust the value of $l_0 = Q$ properly (simple $2l_0 = N/\nu$ may not work). Further implications of this fact are discussed below (Sec. II A).

The torus geometry[40,41], or rectangle (a by b) with periodic boundary conditions[23] (PBC), is characterized by the number of single particle states N_m and aspect ratio $\alpha = a : b$. The area of the rectangle is fixed by $ab = 2\pi\ell_0^2 N_m$[21]. The filling factor is $\nu = N/N_m$ when N electrons are put into the rectangle. The Haldane pseudopotentials can be defined in this geometry, too, albeit they no longer correspond to eigenstates of angular momentum[37].

The rotational symmetry of a sphere, implying the angular momentum $|\vec{L}|$ and its z–component L_z to be good quantum numbers, is replaced by the invariance to magnetic translations in the rectangle with PBC (described in detail by Haldane[42]). The corresponding good quantum numbers are linear momentum along the sides of the rectangle, k_x and k_y. These can take on discrete values[19] $k_x = ik_u$ and $k_y = jk_u\alpha$ with $i, j = 0, \pm1, \ldots, \pm[N/2]$ and $k_u = \sqrt{2\pi/(N_m\alpha)}/\ell_0$ (note that this depends on the filling factor[42]). The magnetic Brillouin zone is therefore rectangular and its size is grows with system size ($\propto \sqrt{N_m}$; Fig. 6 in Ref.[37]).

Conceptually, $|\vec{L}|$ and L_z of the sphere correspond to $|\vec{k}|$ and k_y on the torus. Indeed, the ED spectra from both geometries mapped using $|\vec{L}|/\hbar = |\vec{k}|R$ are in a good quantitative agreement. However, the representations of these symmetries do differ in *finite* systems. The relationship between the orbital degeneracy of a given level and its $|\vec{k}|/k_u$ is non-trivial (nonmonotonous and N_m–dependent), while there are always $2L + 1$ degenerate states for a level with total angular momentum L. Moreover, the orbital degeneracy on the torus, corresponding to rotational symmetry in an infinite system, can easily be lifted by displacing the aspect ratio α slightly from one. We may expect that isotropic states (e.g. a single quasiparticle on the background of an isotropic ground state) will suffer less from the finite size when studied on a sphere. Translationally invariant but anisotropic states (such as a plane wave) will be better served on a torus.

B. Quantum Hall ferromagnets

The basic fact about quantum Hall systems is that in a situation where N electrons have the freedom to occupy N places (single–electron orbitals) out of $2N$, the ground state will be unique and it will possess long–range spin order.

Such a situation typically occurs when two Landau levels are degenerate. The best known example is $\nu = 1$ at zero Zeeman energy. Here, $N = eB/h$ electrons (per unit area) can choose any of $2eB/h$ single-electron states available in the degenerate $0\uparrow$ and $0\downarrow$ Landau levels. The electron-electron interaction implies[43] a unique ground state $|\Psi\rangle$ which is the completely filled $0\uparrow$ LL or any state $R|\Psi\rangle$ where R is an arbitrary rotation of the total spin [leading to an $SU(2)$ symmetry of the GS]. The popular explanation of this effect is the tendency to maximize

the gain in Coulomb exchange energy. All spins in the ground state must be parallel to each other but the direction can be arbitrary. This renders the $\nu = 1$ system to be called a Heisenberg ferromagnet.

Here we investigate another system. A different QHF occurs at $\nu = 2$ when $0\downarrow$ and $1\uparrow$ LL are degenerate as it is the case when the cyclotron energy is equal to the Zeeman splitting. The low lying $0\uparrow$ LL is completely filled and it can be considered inert. The two crossing levels then again dispose of $2eB/h$ single-electron states to be occupied by eB/h electrons. This time, the ground state is twofold degenerate and it consists either of the completely occupied $0\downarrow$ LL or the completely occupied $1\uparrow$ LL (disregarding the occupied $0\uparrow$ LL). The Z_2 symmetry of the GS, regarding the inversion of all spins in the active LLs, earned this system the name Ising QHF.

To make the $0\downarrow$ and $1\uparrow$ LL degenerate, the Zeeman energy has to be adjusted properly to compensate the difference in their Hartree–Fock selfenergies $\Sigma_{n\sigma}$ (unlike $\Sigma_{0\downarrow}$, the selfenergy $\Sigma_{1\uparrow}$ includes exchange with the completely filled $0\uparrow$ LL in addition to the cyclotron energy). Throughout this article we will almost always work at the degeneracy, hence Zeeman energy will be included making energies of the two Z_2 symmetric ground states ($S = 0$ and $S = N/2$, when the occupied $0\uparrow$ LL is included) equal. Experimental techniques to achieve this situation (tilted magnetic field, g–factor reduced by hydrostatic pressure etc.) are summarized elsewhere[37,44].

II. POLARIZATIONS FROM ZERO TO ONE

A. Calculation in terms of electrons

To study the transition between polarized and unpolarized $\nu^* = 2$ ($\nu = 2/3$ and $2/5$) Jain states on a sphere, we have used the series of finite systems $(N, 2Q)$ in which these ground states occur and compared their energy spectra for arbitrary spin configurations. The relation between N and $2Q$ for these Jain states is obtained from the condition of complete filling of two lowest composite fermion (CF) shells at the effective magnetic monopole strength $2Q^* = 2Q - 2(N-1)$. The $\nu = 2/5$ and $2/3$ states occur for $2Q^*$ having the same or opposite sign to $2Q$, respectively[5,6].

Because $g_0^* = 2Q^*+1$ is different from $g_1^* = 2Q^*+3$, the polarized and unpolarized $\nu^* = 2$ states occur in different systems $(N, 2Q)$, corresponding to $N = g_0^* + g_1^*$ or $2g_0^*$, respectively. As an unfortunate artefact of the spherical geometry, the "shift" γ is thus different for the polarized and unpolarized states of electrons, namely $\gamma = 4$ and 3 (at $\nu = 2/5$) and $\gamma = 0$ and 1 (at $\nu = 2/3$).

The fact that the pair of polarized and unpolarized N-electron $\nu^* = 2$ (either $\nu = 2/5$ or $2/3$) states do not occur at the same value of $2Q$ prevents transition (on a sphere) from one to the other through a sequence of spin-flips. Beginning from a polarized state and flipping consecutive $K = N/2 - 1$ spins leads to the system

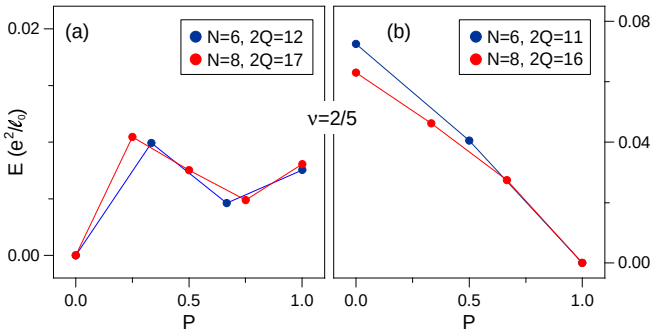

FIG. 1: Excitation energy E as a function of electron spin polarization P, calculated on a sphere for $N = 6$ and 8 electrons, at the values of $2Q$ corresponding to incompressible $\nu = 2/5$ Jain states at $P = 0$ (a) or $P = 1$ (b).

containing two CF's in the $n^* = 1$ LL (so-called quasielectrons, QE's) in addition to the unpolarized state of $N - 2$ electrons. On the other hand, beginning from an unpolarized state and flipping consecutive $K = N/2$ spins leads to the system containing two CF vacances (called quasiholes, QH's) in the polarized state of $N + 2$ electrons.

This discrepancy complicates calculation of the ground state energy E at a fixed filling factor ($\nu = 2/5$ or $2/3$) as a function of spin polarization $P = (N_\uparrow - N_\downarrow)/(N_\uparrow + N_\downarrow) = 2S_z/N$. In contrast to torus geometry, it cannot be simply calculated as the ground state energy for fixed $(N, 2Q)$ as a function of S_z. This forces one into comparison of energies obtained for different N or $2Q$ (problematic in small systems because the energies of the QE's and QH's and of the underlying incompressible Jain state scale differently with N, and because the surface curvature R^{-1} affecting all interaction energies depends on $2Q$).

Therefore, we have calculated separately the energy as a function of S_z for only up to a few spin flips away from the polarized and from the unpolarized $\nu^* = 2$ state (this corresponds to studying the behavior of $E(P)$ separately at $P < 1/2$ and $P > 1/2$, and leaving the $P \sim 1/2$ regime unknown). Recall, however, that the goal is to find the $E(P)$ curve at the Zeeman energy $E_Z = E_Z'$ for which the polarized and unpolarized Jain states are degenerate, $E(0) = E(1)$. Hence, in order to estimate E_Z', one has to know the energies of both polarized and unpolarized Jain states corresponding to the same $(N, 2Q)$. In other words, one needs the estimate of E/N (energy per electron) in both polarized and unpolarized Jain state in small systems.

Let us explain how it is done on the example of $\nu = 2/5$ and $N = 8$. To get $E(P)$ at small P we use finite-size calculation of $E(S_z)$ starting with the unpolarized Jain state at $2Q = 17$. In the CF picture of this state, $2Q^* = 3$, $g_0^* = 4$, and the $N = 8$ CF's fill completely the $0\uparrow$ and $0\downarrow$ LL's. We calculate $E(S_z)$ in the whole range of S_z. At $S_z = N/2$, the ground state contains two QH's in the

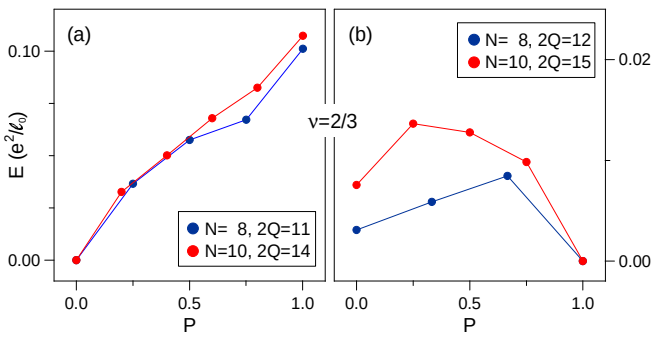

FIG. 2: Same as Fig. 1, but $\nu = 2/3$ and $N = 8, 10$.

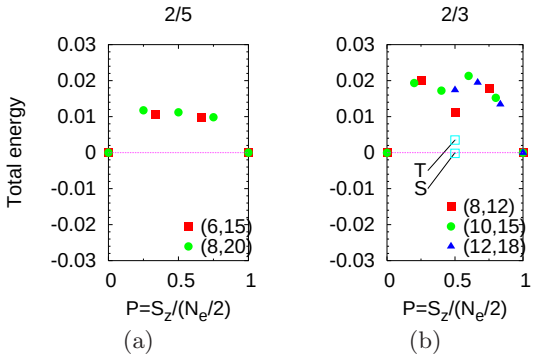

FIG. 3: $E(P)$ of the $\frac{2}{5}$ and $\frac{2}{3}$ systems on a torus. Here, the singlet and polarized states occur in systems with the same (N, N_m), so that the $E(P)$ dependence shown here is relevant in the whole range of P. The $1/N \to 0$ extrapolated values are displayed by 'S' (sphere) and 'T' (torus).

polarized Jain state, whose energy must be subtracted to find E'_Z correctly. This is done easily by replacing the calculated $E(S_z = N/2)$ by the energy obtained for the polarized Jain state at $2Q = 16$, rescaled appropriately by $\sqrt{16/17}$ to account for a different ℓ_0. With such estimate of E'_Z we ignore all but the few values of $E(S_z)$ corresponding to the smallest S_z, and recalculate them into $E(P)$ at small P. The result is plotted in Fig. 1a, also showing weak finite-size effects (due to the $g_0^* \neq g_1^*$ discrepancy) at small P, confirmed by comparison with the $N = 6$ calculation. The calculation of $E(P)$ at large P goes analogously, starting with the polarized Jain state at $2Q = 16$, and with the energy of the unpolarized Jain state needed for E'_Z obtained by rescaling the value at $2Q = 17$ and $S_z = 0$. Again, as shown in Fig. 1b, comparison of data for $N = 6$ and 8 confirms the size-convergence. The same procedure has been carried out for $\nu = 2/3$, with the results plotted in Fig. 2a,b.

While the calculation on a sphere allowed us to find $E(P)$ only at small or large P (and with some uncertainty in the estimate of E'_Z), one conclusion seems established despite finite-size problems: $E(P)$ at E'_Z increases when P is either increased from 0 or decreased from 1. In other words, $E(P)$ at small either P or $1 - P$ is larger

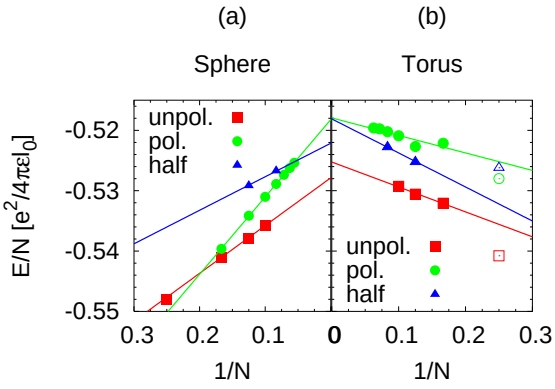

FIG. 4: Energy per particle of the singlet state, the HPS and the fully polarized state at $\nu = \frac{2}{3}$. Different system sizes are shown ($N = 4$ to 18 electrons), extrapolation to infinite systems, $1/N \to 0$ (only solid points were used).

than $E(0) = E(1)$, which would imply abrupt transition between the polarized and unpolarized $n^* = 2$ Jain states as a function of E_Z.

Contrary to the sphere, it is possible to scan the whole range of P using calculations on a torus (square with periodic boundary conditions). Choosing (N, N_m), the number of electrons and magnetic flux quanta, the sector of spin S_z corresponds to filling factor $\nu = N/N_m$ and polarization $P = S_z/(N/2)$.

The principal conclusion is to confirm the observations made for a sphere: all states with intermediate polarization $0 < P < 1$ are higher in energy than the polarized ($P = 1$) and the singlet state ($P = 0$). Systems of different sizes exhibit qualitatively the same behaviour both at filling factor $\frac{2}{5}$ and $\frac{2}{3}$, Fig. 3. On a quantitative level and within the small systems accessible to exact diagonalization, it seems unlikely that this situation changes if we continue to larger systems. Nevertheless, the ground state energies at $P = 0, \frac{1}{2}$ and 1 extrapolated to $1/N \to 0$, Fig. 4, give opposite predictions in this aspect: $E(\frac{1}{2}) > E(0) = E(1)$ on a torus while the oposite was found on the sphere[33]. It is, however, important to note that the extrapolation for $E(\frac{1}{2})$ is based only on two points. Moreover, it cannot be established reliably whether all these points correspond to a realisation of the same state in an infinite system[37]. On the other hand, it is possible to speculate that the $P = \frac{1}{2}$ state of $\nu = \frac{2}{3}$ is more stable than other states with $|P - \frac{1}{2}| < \frac{1}{2}$, i.e. that there may be a downward cusp at $P = \frac{1}{2}$, Fig. 3b.

B. Composite fermion calculation

In the composite fermion (CF) picture[7], the $\nu = 2/3$ and 2/5 fillings correspond to the same effective CF filling factor $\nu^* = 2$. The completely polarized ($S = N/2$ and $P = 1$) and unpolarized ($S = P = 0$) states are represented by simple single-particle configurations, with a

pair of completely filled CF LL's: 0↑ and either 1↑ or 0↓. Stability of the corresponding four Jain states ($\nu = 2/3$ or $2/5$; $P = 0$ or 1) requires a splitting Δ of the 1↑ and 0↓ LL's, and their incompressibility is attributed to a single-particle (cyclotron) CF gap $\hbar\omega_c^*$.

The intermediate polarizations ($0 < P < 1$) are possible for nearly degenerate CF self-energies $\Sigma_{n\sigma}$ of the 1↑ and 0↓ LL's (Sec. I.B). Whether such partially polarized states indeed occur for some appropriate Δ depends on the CF–CF interactions within and between the two partially filled CF LL's. If they do, their many-CF wavefunctions, the CF–CF correlations, and possible incompressibility (at least at some of the intermediate values of P) also depends completely on the CF–CF interaction. Remarkably, the $P = 1$ to 0 (paramagnet-ferromagnet) transition occurs directly for the electrons filling two LL's ($\nu = 2$)[8], but partially polarized states were suggested in a mixed electron–CF system at $\nu = 4/3$[38].

Interaction between two particles (e.g., electrons or CF's) in a pair of LL's (n_1, σ_1) and (n_2, σ_2) is determined by Haldane pseudopotential $V(\mathcal{R})$, defined as pair interaction energy as a function of relative angular momentum[4]. For a pair of identical CF's in the same LL (here, 0↓ or 1↑), the allowed $\mathcal{R}$'s are odd integers. For two CF's distinguished by spin and/or LL index, $\mathcal{R}$ can be odd or even. Assigning pseudospins ↑ and ↓ to the CF LL's 1↑ and 0↓, the CF dynamics within these two levels is determined by a set of three pseudopotentials: $V_{\uparrow\uparrow}(\mathcal{R})$ and $V_{\downarrow\downarrow}(\mathcal{R})$ for $\mathcal{R} = 1, 3, 5, \ldots$, and $V_{\uparrow\downarrow}(\mathcal{R})$ for $\mathcal{R} = 0, 1, 2, \ldots$. Because $V_{\uparrow\uparrow}$ and $V_{\downarrow\downarrow}$ describe interaction in different LL's, they are not equal. Hence, the CF–CF interaction within these two LL's is pseudospin-asymmetric.

Knowing V is the key to understanding the CF dynamics in partially filled shells. The correlations have particularly simple form when $V(\mathcal{R})$ is dominated by one coefficient. In this case, the particles interacting through $V(\mathcal{R})$ tend to avoid the corresponding high-energy pair state[4]. More generally, the correlations depend only on the anharmonic contribution to $V(\mathcal{R})$, where the harmonic dependence means V linear in average squared distance $\langle r^2 \rangle$, which corresponds to a roughly linear $V(\mathcal{R})$[9].

The three CF–CF pseudopotentials, $V_{\uparrow\uparrow}$, $V_{\downarrow\downarrow}$, and $V_{\uparrow\downarrow}$, are known quite well for electron systems near $\nu = 1/3$, where they describe interactions among Laughlin QE's[10] or reversed-spin QE_R's[11,12]. At long range (large $\mathcal{R}$) these pseudopotentials must be consistent with Coulomb repulsion of two fractional charges, $-e/3$. At short range they can be obtained from finite-size calculations[13,14,15,16] and show features revealing the CF internal structure. Radial charge distributions of QE and QE_R's are presented in Fig. 5a. They were calculated numerically from the exact eigenstates of 10 electrons, and they are normalized to $\int \varrho(r) r dr = 1/3$ (in the length units of ℓ_0). Comparison with the electron charge profiles plotted in the inset shows that, except for the reduced QE/QE_R charge, the CF's and electrons in their lowest LL's are very similar (QE_R and LL_0 in Fig. 5a,b), while

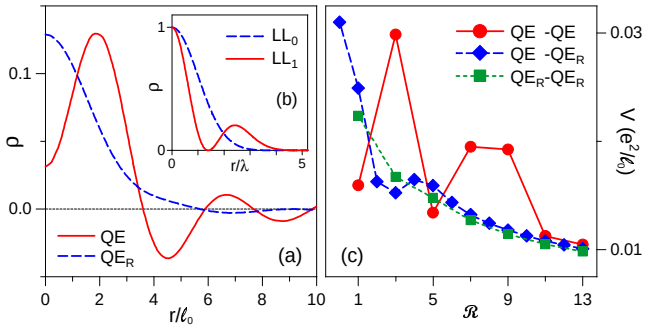

FIG. 5: (a) Radial charge distributions of QE_R and QE at $\nu = 1/3$, i.e., of the CF's in the lowest and first excited LL. (b) Same for the electrons. (c) CF–CF interaction pseudopotentials at $\nu = 1/3$.

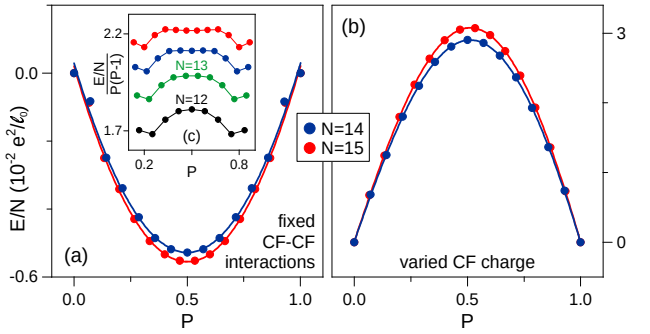

FIG. 6: (a) Energy per particle, $\varepsilon = E/N$, as a function of CF spin polarization P, calculated for $N = 14$ and 15 CF's at $\nu^* = 2$ using effective CF–CF interactions of Fig. 5; curves – parabolic fits. (b) Anharmonic coefficient $\alpha/4 = \varepsilon/P(P-1)$ for $12 \le N \le 14$ (same units). (c) Same as (a) but using fill-dependent CF–CF interactions corresponding to a variable CF charge (see text); curves – polynomial fit.

the CF's and electrons in their excited LL's are quite different (QE and LL_1 in Fig. 5a,b). In Fig. 5(c) we plot the QE–QE, QE–QE_R, and QE_R–QE_R pseudopotentials obtained by combining the short-range data from exact diagonalization and the electron–electron parameters at long range. These are the effective interactions that we used in numerics at $\nu^* = 2$. We also removed the (artificial) discrepancy between the degeneracy of 0↑ and 0↓ CF LL's on a sphere by considering a pair of LL shells with the same angular momentum l.

The computation consisted of the exact diagonalization of the $V = [V_{\uparrow\uparrow}, V_{\downarrow\downarrow}, V_{\uparrow\downarrow}]$ interaction hamiltonian, separately for each combination of $N_\uparrow$ and $N_\downarrow$ (CF numbers in the two LL's) giving $N \equiv N_\uparrow + N_\downarrow = 2l + 1$, i.e., corresponding to $\nu^* = 2$. Note that because the length of pseudospin is not conserved by V, a separate diagonalization is required for each pseudospin projection $S_z = (N_\uparrow - N_\downarrow)/2$, corresponding to different polarizations $P = 2S_z/N$ of the CF system [or $(1 + P)/2$ of the whole electron state].

The result is the dependence of energy on polariza-

tion, $E(P)$, tilted (by adding the appropriate linear Zeeman term $E'_Z S_z \propto P$) to the $E(0) = E(1)$ situation. In Fig. 6a we compare the data for two largest CF numbers we have used, $N = 14$ and 15. For only one spin flip away from $P = 0$ or 1, the ground state at each N is a single spin-wave. For more than one spin flip, regular dependence of energy on $K/N \sim P$ rather than on K is evident. In this ($P \sim 1/2$) regime, E at each P scales roughly linearly with N, and the excitation energy per particle $\varepsilon(P) = E(K/N)/N$ becomes a convergent characteristic of the macroscopic system. The convergence with increasing the system size is evident, with a continuous $E(P)$ curve emerging for $N \to \infty$. The data are polarization-symmetric, $E(P) = E(1-P)$, reflecting the particle–hole symmetry at a half-filling of a pair of shells (note that $E(0) = E(1)$ is equivalent to $\Sigma_{0\uparrow} = \Sigma_{1\downarrow}$).

The $\varepsilon(P)$ curve determines dependence of the ground state polarization P on the Zeeman gap in tilted-field experiments. To calculate $P(E_Z)$, one must find the minimum of the total energy (per particle) including a linear Zeeman term, $\varepsilon(P) - P(E_Z - E'_Z)/2$. Clearly, only the convex points of $\varepsilon(P)$ can become ground states at the appropriate E_Z. A special case is a convex parabola, $\varepsilon(P) = \alpha/4 \cdot P(P-1)$, leading to a linear dependence, $P(E_Z) = 1/2 + (E_Z - E'_Z)/\alpha$, with P varying between 0 and 1 over the E_Z range of length α.

Fig. 6a show that $\varepsilon(P)$ indeed is nearly parabolic, so in Fig. 6a we plotted $\alpha(P) = \varepsilon(P)/P(P-1)$ to study the anharmonic contribution. Only the result for $N = 12$ (α having a local maximum at $P = 1/2$) agrees with the earlier calculation[20], also showing a downward cusp of $\varepsilon(P)$ at the half-polarization. Such cusp would lead to an inflexion or a plateau in $P(E_Z)$ around E'_Z.

The emergence of a plateau would imply that the system is not affected by infinitesimal variation of the gap Δ, i.e., that it is incompressible and should exhibit quantum Hall effect. However, our calculations for larger systems seem to invalidate the prediction of a plateau, showing disappearance of the downward cusp in $\varepsilon(P)$ for $N > 12$. In Fig. 6c this is seen as transition from a local maximum to a local minimum in $\alpha(P)$ at $P = 1/2$. Remarkably, in experiment, the partially polarized states were only observed over a narrow polarization range around $P = 1/2$, implying a well developed plateau in $P(E_Z)$, in disagreement with the CF calculation.

C. Composite fermions with fill-dependent charge

We found notable qualitative disagreement between the numerical results obtained (i) in terms of CF's at $\nu^* = 2$ and (ii) in terms of electrons at $\nu = 2/3$ or 2/5. The first approach allows for studying fairly large systems and is free of the troubling artificial $g_0^* \neq g_1^*$ asymmetry on a sphere. However, the results obtained using the latter, more direct approach appear more consistent in both used geometries. The main conclusion, too, seems established despite finite-size effects: the absence of a ground state of intermediate polarization between $P = 0$ and 1. While the experiment[28] indicates a stable half-polarized quantum Hall state in apparent contradiction with approach (i), the (suggested earlier[20]) agreement with approach (ii) is also not convincing in view of our numerics for larger systems and different geometries.

The most questionable assumption in using the CF model is that the interactions among the CF's at $\nu^* = 2$ can be described by a set of three two-body pseudopotentials, independent of the filling of $0\uparrow$ and $0\downarrow$ CF LL's. Consequently, these pseudopotentials are estimated at $\nu = 1/3$, for the QE–QE, QE–$\mathrm{QE_R}$, and $\mathrm{QE_R}$–$\mathrm{QE_R}$ pairs (i.e., with only two CF's present in the $0\uparrow$ or $0\downarrow$ LL). Such approach was proven successful only for polarized systems with QE fillings merely up to $\nu_{\mathrm{QE}} = 1/3$ (corresponding to $1/3 \leq \nu \leq 4/11$)[17]. On the other hand it is well-known that the form of an actual electron excitation represented by a CF depends on the filling factor. For example, charge of a Laughlin QE at $\nu = 1/3$ is $-e/3$, while charge of QH at $\nu = 2/5$ is only $e/5$.

This implies (significant) reduction of all three CF–CF pseudopotentials when going from $\nu = 1/3$ to 2/5, demonstrated earlier for polarized systems[14,15]. Clearly, the pseudopotentials determined at $\nu = 1/3$ cannot be used at $\nu = 2/5$ with great confidence (note, however, that we have checked that the results are quite insensitive to the model V's used, as long as they retain qualitative behavior at short range). But more importantly, it probably also invalidates the concept of using fixed two-body pseudopotentials $V_{\uparrow\uparrow}$, $V_{\downarrow\downarrow}$, and $V_{\uparrow\downarrow}$, which are independent of the filling of each of the two CF LL's (at least in the whole range between the empty and full shell). While the electron system at $\nu = 2/3$ or 2/5 may well be correctly represented by a two-pseudospin fluid of CF's with two-body forces, the polarization-dependence of the effective CF–CF pseudopotentials must probably be taken into account when modelling the $P \sim 1/2$ regime.

As a test, we allowed for a very simple dependence of the CF–CF interactions on P. We assummed a linear dependence of the charge q_σ carried by a CF with pseudospin $\sigma = \uparrow$ or $\downarrow$ on the partial filling ν_σ of its LL (with $q_\sigma = 1/3$ and 1/5 at $N = 0$ and g^*, respectively). For interaction pseudopotentials we took $V^P_{\sigma\sigma'} = q_\sigma q_{\sigma'} V_{\sigma\sigma'}$, with $V_{\sigma\sigma'}$ shown in Fig. 5c. The assumption that only the scale of V depends on the LL filling (with the structure unaffected) is justified by the comparison of V in polarized $\nu = 1/3$ and 2/5 states[14].

The $E(K)$ calculated in this way and plotted in Fig. 6b shows opposite (concave vs. convex) behavior to Fig. 6a obtained ignoring fill-dependence of V. Again, the values at only a few spin flips away from $P = 0$ or 1 scale best with N and K, but a convergent $\varepsilon(P) = E(K/N)/N$ curve emerges around $P = 1/2$. Note that though the CF particle-hole pairs become charged for $q_\uparrow \neq q_\downarrow$, this artefact does not affect the interesting regimes of $P = 0$ or 1 (exactly) or $P \sim 1/2$.

The contrast between $E(K)$ shown in Figures 6a and b is an obvious warning that the CF–CF interactions used

so far to model $\nu^* = 2$ may have been not exact enough. To the best of our knowledge, the fact that quasiparticles (of e.g. $\nu = 2/5$) around $P = 0$ and around $P = 1$ might have different charge and hence different interactions, has not been previously considered. Although the exact particular form of the interaction, $V^P_{\sigma\sigma'}$, chosen in Fig. 6b, does not describe correctly $\nu^* = 2$ in the whole range of P, it does show that models of CF–CF interactions used in earlier works (assuming all quasiparticles to have the same charge for all values of P) may have given qualitatively incorrect predictions. In order to determine whether $E(\frac{1}{2})$ is larger or smaller than $E(0) = E(1)$, correct CF–CF interaction must be found.

III. POLARIZATION ONE HALF

Owing to the Z_2–symmetry of the ground state (GS), it is customary to call the systems at $\nu = 2$ an Ising–type quantum Hall ferromagnet[34] (QHF). Regarding only the ground state, the $\nu^* = 2$ ($\nu = \frac{2}{5}, \frac{2}{3}$) systems fall into the same category. Despite this, there are substantial differences between the integer and fractional systems. Most importantly, the possible onset of domain formation in the integer systems is replaced by an antiferromagnetic ordering in the systems at fractional filling.

When the two ferromagnetic GS's are degenerate, the complete spectrum of the $\nu = 2$ system, Fig. 7b, is symmetric under the spin inversion. While Heisenberg ferromagnets, have a spin-wave with vanishing energy at $k \to 0$ as the lowest excitation, Fig. 7a, the first excited state of a $\nu = 2$ system has a single spin flip at a finite wavevector k, Fig. 7b. This is in line with Goldstone theorem which requires a continuous symmetry of the GS, which is SU(2) in the former case.

The corresponding full spectra of the $\nu^* = 2$ systems, Fig. 8, do not have any obvious structure resembling the one of $\nu = 2$, Fig. 7b. The spin–inversion symmetry is missing, $S_z = 3.0$ and $S_z = 1.0$ states have different energies, Fig. 8. No definite prediction can be made about the spin of the lowest excitation. Perhaps most importantly, the $\nu = \frac{2}{3}$ and $\frac{2}{5}$ spectra look very differently, Fig. 8a,b, except for the GS at $k = 0$ and its gap. This is markedly at odds with the picture of non-interacting composite fermions according to which the spectra should be the same after rescaling to equal effective magnetic length.

Let us now concentrate on the sector of S_z corresponding to equal number of up and down spins in the active Landau levels. For $\nu = \frac{2}{3}, \frac{2}{5}$ using the ED with electrons this means $S_z = N/4$ and for $\nu = 2$ with the low–lying $0\downarrow$ level neglected, it is $S_z = 0$.

The spectrum of a $\nu = 2$ system has a clear structure, Fig. 9a. A $(2N-2)$–tuplet of states distinguished by

$$\vec{k}_r/k_u = (0, \pm i) \text{ or } (\pm i, 0)\,, \quad i = 0, 1, 2, \ldots, N/2, \quad (1)$$

is separated from higher excited states. Rezayi *et al.*[34]

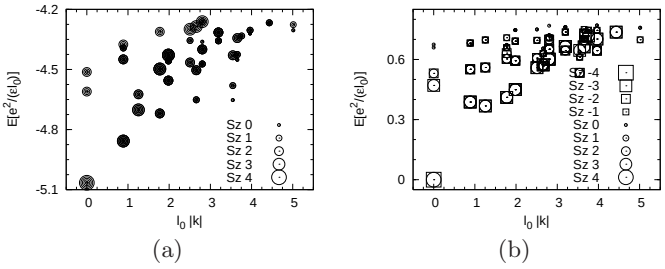

FIG. 7: Spectra of (a) $\nu = 1$ and (b) $\nu = 2$ quantum Hall ferromagnets. Eight electrons on a torus. The complete degeneracy of $S_z = -S, -S+1, \ldots, S$ levels (a) corresponds to a Heisenberg ferromagnet, the degeneracy of S_z and $-S_z$ only (b) is proper to an Ising ferromagnet.

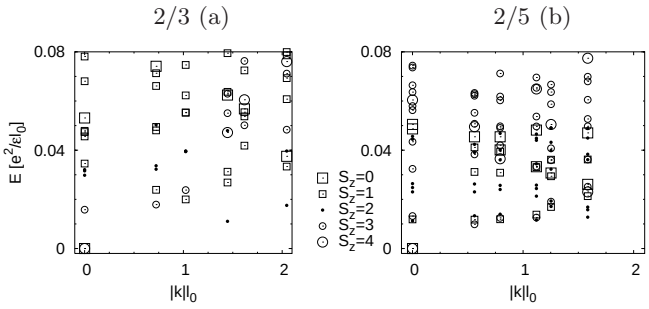

FIG. 8: Full spectrum of eight electrons at $\nu = \frac{2}{3}$ and $\frac{2}{5}$ (hence both $\nu^* = 2$) on a torus with Zeeman energy adjusted so that the fully polarized and spin-singlet incompressible states are degenerate. To be compared with $\nu = 2$, Fig. 7b.

identified this group as a state with two (Ising) domains in different system of the same type.

The easiest way to see this is in the spin-resolved density-density correlation functions[45]. Another possibility[34] is to replace the square in our model by a rectangle with periodic boundary conditions while keeping its area fixed, Fig. 10. A group of N states quickly detaches from the $(2N-2)$-tuplet, once the aspect ratio $\alpha = a : b$ of the rectangle exceeds ≈ 1.2. These states have $\vec{k}_r/k_u = (\pm i, 0)$, $i = 0, 1, 2, \ldots, N/2$, their degeneracy improves with increasing α and the energy changes roughly proportional to $1/\sqrt{\alpha} \propto b$. It is very suggestive that the change of energy is mostly due to changing length of the domain walls which are likely to be oriented along the shorter side of the rectangle.

This method of investigation is particularly useful for small systems. It is not possible to distinguish the $(2N-2)$-tuplet in a $N = 6$ system ($\nu = 2$), because its degeneracy is far from being perfect, so that it is mixed up with higher excited states, Fig. 9b. A minute variation of the aspect ratio, however, separates the N-tuplet of states with domains oriented parallel to b, Fig. 10a. The energy cost of a domain wall per magnetic length obtained for both system sizes in Fig. 10 is the same,

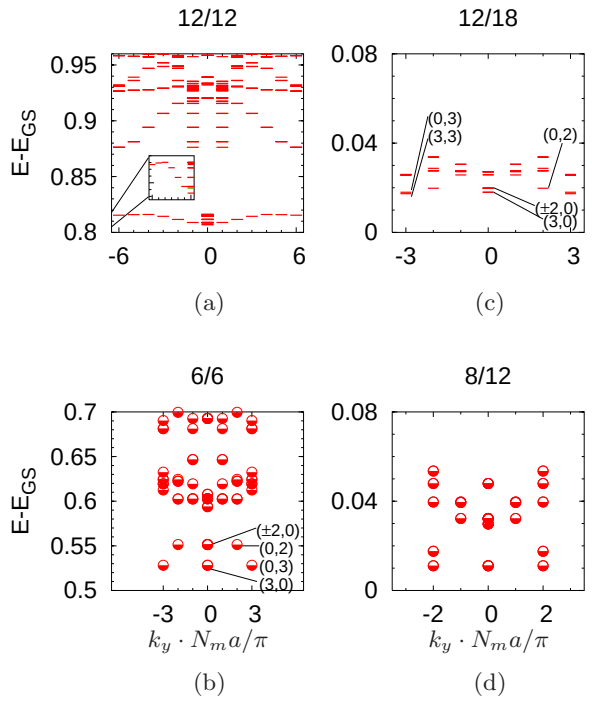

FIG. 9: Spectra of the half–polarized sector in a $\frac{2}{3}$ and $\nu = 2$ systems. The titles give N/N_m, number of electrons N and filling factor $\nu = N/N_m$. The reference energy is the ferromagnetic ground state at degeneracy. Small numbers at some of the states are their linear momenta $(k_x, k_y)/k_u$.

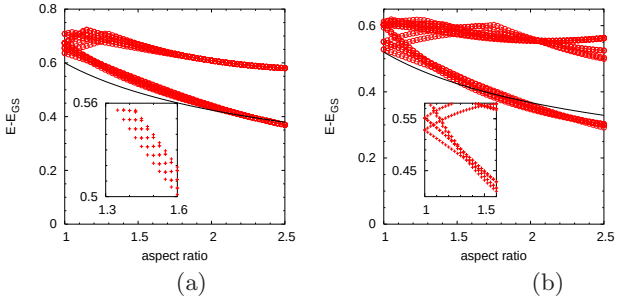

FIG. 10: Energies of the $S_z = 0$ states in the $\nu = 2$ Ising ferromagnet relative to the ground state ($S_z = \pm N/2$). The solid line is a fit $c/\sqrt{\alpha}$, $\sqrt{\alpha} \propto b$. Right $N = 6$ ($c = 0.52$), left $N = 8$ ($c = 0.60$).

$0.042e^2/4\pi\varepsilon\ell_0$.

The $\nu = \frac{2}{3}$ system has again a rather different spectrum of the $S_z = N/4$ sector, Fig. 9c,d compared to Fig. 9a,b. No similar grouping of states is obvious. On the other hand, because six electrons should be in the inactive CF LL in the $N = 12$ system for instance, we should also keep in mind a comparison between the $(N, N_m) = (12, 18)$ and $(6, 6)$ spectra, Fig. 9c and 9b. In both systems, the lowest energy states form a loose group (the marked states in Fig. 9bc). This group is separated

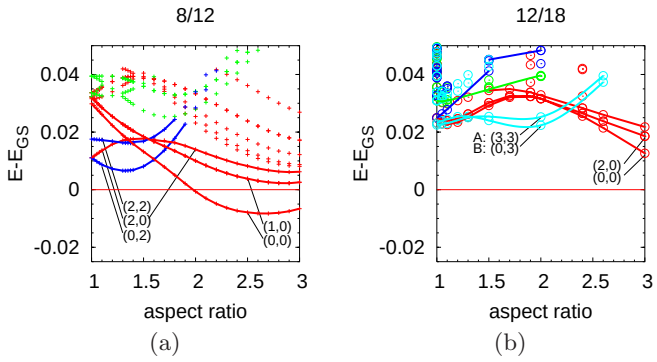

FIG. 11: Energies of the HPS under varying aspect ratio (Coulomb interaction, $\frac{2}{3}$). (a) $N = 8$, (b) $N = 12$.

from other states by 0.01 (Fig. 9c) and $0.04e^2/4\pi\varepsilon\ell_0$ (Fig. 9b) what is slightly more than differences between energies within the group. The states can be classified by their momentum $\vec{k}_r/k_u$: these are $(\pm 2, 0), (3, 0)$ for $\nu = 2$ and $(\pm 2, 0), (3, 0), (3, 3)$ for $\nu = \frac{2}{3}$ plus their $x - y$ symmetric states. For $\nu = 2$, these states belong to the $(2N - 2)$–tuplet of the single–domain state.

The $\vec{k}_r/k_u = (3, 3)$ state of $\nu = \frac{2}{3}$, Fig. 9c, as the only clear difference between Fig. 9c and b, cannot be just a finite size artefact. Under a slight squeeze, the $\frac{2}{3}$ systems reveal a clearly different behaviour compared to $\nu = 2$. The $\vec{k}_r/k_u = (3, 3)$ state (marked by A in Fig. 11b) quickly becomes the absolute ground state of the system, together with the $\vec{k}_r/k_u = (0, 3)$ state (B in Fig. 11b). These two states react to the squeezing very similarly within the range $1.3 < \alpha < 2.3$. With some experience from $\nu = 2$ systems, Fig. 10, this range of α may correspond to the lifting of the $x - y$ degeneracy while still preserving the 2D character of the system (α not too far from one). An N-tuplet similar to the integer filling systems (Eq. 1) does not appear as far as for $\alpha < 3$. Even though such grouping is possible for larger aspect ratios, their eventual relevance would have to be supported by some strong external anisotropy justifying the large aspect ratio chosen for the model.

Going from smaller to larger systems, Fig. 11a and 11b, it seems indeed possible that the two states, $\vec{k}_r/k_u = (0, N/2), (N/2, N/2)$ become the lowest states with $S_z = N/4$ when the $x-y$ symmetry is lifted. Namely, $\vec{k}_r/k_u = (0, 2), (2, 2)$ for $N = 8$ and $\vec{k}_r/k_u = (0, 3), (3, 3)$ for $N = 12$ do and $k_x/k_u = 2$ and 3 are the maximal k_x-values in the finite system with $N = 8$ and 12 electrons.

Albeit distinguished by $\vec{k}_r$, the two states (A,B in Fig. 11b) look very similar in their spin–resolved density–density correlation functions

$$g_{\uparrow\uparrow}(\vec{r}) = \langle \delta(\vec{r}_1 - \vec{r}_2 - \vec{r}) \delta_{\sigma_1\uparrow} \delta_{\sigma_2\uparrow} \rangle \,,$$

as shown in Fig. 12c,e. Other combinations of spins not shown in Fig. 12 ($\downarrow\downarrow$, $\uparrow\downarrow$) also confirm this conclusion.

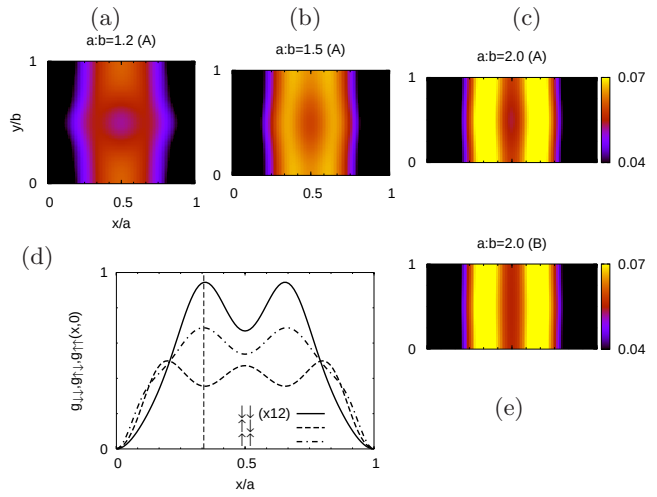

FIG. 12: Correlation functions $g(\vec{r}) = g(x, y)$ of the $\vec{k}_r/k_u = (3, 3)$ half-polarized state (A in Fig. 11b). Under slight variation of the aspect ratio $\alpha = a : b$, the conditional probability $g_{\uparrow\uparrow}(\vec{r})$ indicates antiferromagnetic ordering (a,b,c), confirmed by $g_{\downarrow\downarrow}$ and $g_{\uparrow\downarrow}$, (d). The state B from Fig. 11b has a very similar structure (e).

We observe for both states two vertical stripes (maxima) in $g_{\uparrow\uparrow}(\vec{r})$, $g_{\downarrow\downarrow}(\vec{r})$ together with the two complementary stripes (minima) in $g_{\uparrow\downarrow}(\vec{r})$, Fig. 12d. This could mean that six electrons in the active CF Landau levels align antiferromagnetically, $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$ following the elongated side of the elementary cell.

IV. DISCUSSION AND SUMMARY

With near degeneracy of $0\downarrow$ and $1\uparrow$ CF LL's, quantum Hall ferromagnet of $\nu^* = 2$ is a unique system: (i) two different incompressible CF liquids form for P=0 and 1; (ii) low-energy excitations of both liquids involve spin; (iii) experiment suggests another liquid at P=1/2; (iv) in the P=1/2 liquid interactions among CF's play crucial role (unlike at P=0 or 1 where incompressibility is due to LL filling and the interactions are not important). All together, (v) the concept of incompressible states of correlated CF's at these fillings appears even more fascinating than in the states discovered more recently at $\nu = 4/11$ or $3/8$[18], due to additional spin freedom.

The comparison of the experimental evidence with our various numerical calculations demonstrates the lack of understanding of the microscopic origin of the half-polarized quantum Hall states. Yet, it appears very difficult to model these states in finite-size numerics.

In the electron calculation, large Hamiltonian dimensions make exact calculation of the $P = 0$ state very complicated already for $N > 8$. For $N = 8$ only two spin flips separate $P = 0$ or 1 from $P = 1/2$ (the first one being simply a spin-wave), which might not be enough to capture physics of the correlated $P \sim 1/2$ regime. Moreover, calculations on a sphere suffer from the $g_0^* \neq g_1^*$ artefact that further complicates interpretation of the results in this geometry.

In the CF calculation, the result strongly depends on the choice of effective CF–CF interactions, which are not known with near enough accuracy. This problem does not appear in the understanding of Jain states corresponding to filled CF LL's (provided these interactions are weaker than $\hbar\omega_c^*$), but here it is essential. In the CF picture, one particle or hole in a CF LL represents different electronic excitations depending on the filling of a CF shell. These excitations are only known in some special cases, when they correspond to e.g., Laughlin QE's or $\mathrm{QE_R}$'s (in an empty CF LL) or Jain QH's (in a full CF LL). Consequently, although it seems plausible that the low-energy dynamics of the electron states corresponding to partially filled CF LL's is generally well described by two-body effective CF–CF interactions, their pseudopotentials are not well known.

It seems that electron calculations in larger systems (preferably in toroidal geometry) are needed for understanding of the occurrence and incompressibility of half-polarized $\nu = 2/5$ and $2/3$ states. More advanced exact diagonalization as well as Monte Carlo methods must be considered. On the other hand, further experimental studies are much needed in view of possible insight into the nature of CF–CF interactions.

In summary, at the level of present computational capacity, the calculations for toroidal *and* spherical geometry in Sec. IIA indicate that the ferromagnet-paramagnet transitions both at $\nu = 2/3$ and $2/5$ are abrupt. This applies to homogeneous and isotropic systems. The antiferromagnetically ordered states at polarization one half (Sec. III) could in principle, however, become the absolute ground state near the transition if a suitable anisotropy or inhomogeneity in the system is present. These states also constitute probably the best demonstration of marked differences between the fractional and integer QHF. In the latter case, the system splits into two equally large domains ($\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow$).

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