Transport in quantum systems

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1 Preliminaries

1.1 Heterostructures and quantum wells

Quasi-2D electron gases (2DEGs) can arise on an interface between semiconductors due to size quantization (subbands). Whether the quasi-2D character of such a system can be detected or not depends most importantly on the distance between the quantized energy levels E_n and the total concentration of carriers in the 2DEG.

The quantized energy levels are found by solving the Schrödinger equation

$$\frac{-\hbar^2}{2m^*}\psi_n''(z) + V(z)\psi_n(z) = E_n\psi_n(z) \tag{1}$$

in which the total electric potential $V(z)$ comprises of several parts. The conduction band edge profile $V_b(z)$ and electric field contributions due to ionized donors $V_d(z)$ are relatively simple to treat; $V(\vec{r}) = V(z)$ however also depends on the electron (charge) density $q(\vec{r}) = e\rho(z)n_{2DEG}$

$$\Delta V_e(\vec{r}) = \frac{q(\vec{r})}{\epsilon} \tag{2}$$

that is determined by the (occupied) states $\psi_n(z)$ according to $\rho(z) = \sum_n \nu_i|\psi_n(z)|^2$; n_{2DEG} is the areal density of electrons in each subband (related to the 2D density of states) and m^* is the electron effective mass. Correlations due to Pauli principle (two electrons with the same spin cannot be at the same spot at the same time) are accounted for using a *local* exchange term $V_x(z) = -1.19(e^2/4\pi\epsilon)(3\rho(z)n_{2DEG}/4\pi)^{1/3}$ (this is called the local density approximation). This approximative approach largely ignores the many-body character of the wavefunction describing many (typically 10^{11} per square centimetre) electrons and emerges from Hartree-Fock perturbative evaluation of electron-electron interaction in a 3D electron gas (p. 336 in [1]). The system of equations (1,2) with $V = V_b + V_d + V_e + V_x$ is typically solved iteratively and resulting subband fractional occupations ν_i are called *self-consistent solution to Poisson and Schrödinger equations.*

Depending on the form of V_b and doping (V_d), heterostructures, quantum wells or superlattices can be described. Eq. (1) is derived (see Sec. 2.2 of Winkler's book [35]) within the *envelope function approximation* applied to the conduction band in case it is described by a single effective mass m^* (commonly, this is the Γ_6 band in zinc-blende direct-band semiconductors such as GaAs). Any of the functions $\psi_n(z)$ should be understood as an amplitude (e.g. within some tight-binding model) that modulates the Bloch-periodic part of the wavefunction, i.e. the replacement for the first factor in $\Psi(z) = \exp(ikz)u_k(z)$ (such Ψ would apply to an infinite periodic system while Eq. 1 is meant to describe e.g. a heterostructure). Alternatively, the following product ansatz can be considered:

$$\Psi(x, y, z) = \exp(ixk_x + iyk_y)\psi(z)u_{\vec{k}}(z)$$

where $u(z) = u(z+a)$ is the Bloch-periodic part (with lattice constant a) and $\psi(z)$ is slowly-varying for $z/a \sim 1$ (*viz* envelope).

3D	$\frac{(2m)^{3/2}}{4\pi^2\hbar^3}(\varepsilon-\varepsilon_0)^{1/2}$	$k_F^3/(6\pi^2)$
2D	$\frac{m}{2\pi\hbar^2}$	$k_F^2/(4\pi)$
1D	$\frac{(2m)^{1/2}}{2\pi\hbar}(\varepsilon-\varepsilon_0)^{-1/2}$	k_F/π

Table 1: Density of states $g(\varepsilon)$ of free electrons (no spin degeneracy) and electron density n.

	n [cm^{-2}]	μ [cm^2/Vs]	
ZnO/ZnMgO	6×10^{11}	5500	
liquid He	10^5-10^9	$\sim10^6$	[30]
Si/H	6.8×10^{11}	17×10^3	
MoS_2		0.48×10^3	
graphene	2×10^{11}	0.2×10^6	
GaAs/GaAlAs	3×10^{11}	33×10^6	[28, 29]

Table 2: Mobilities in 2D systems.

1.2 Density of states

For a given multiband dispersion, $E=E_i(\vec{k})$, the density of states (DOS) counts the number of states per small energy interval around energy ε. Because we will now often speak about infinite systems (in d dimensions) — where states are indexed by the wavevector $\vec{k}$ — DOS is the density in energy per unit volume and its natural units in solid-state physics are $\mathrm{eV}^{-1}\mathrm{nm}^{-d}$. Apart from the canonical definition $g(\varepsilon)=V^{-1}dN/d\varepsilon$ (N is the number of occupied states with energy less than ε and we took 3D case for definiteness), there are two other useful equivalent definitions:

$$g(\varepsilon)=\sum_i\int\frac{d^3k}{(2\pi)^3}\delta\big(\varepsilon-E_i(\vec{k})\big)\equiv\sum_i\int_{FS}\frac{d\Omega}{(2\pi)^3}k^3/|\vec{k}\cdot\nabla_kE_i(\vec{k})|^{-1}\,. \tag{3}$$

The second expression uses only a double integral over the i-th Fermi surface $\varepsilon_F=E_i(\vec{k})$.

Assuming free electrons in single band, $E(\vec{k})=\hbar^2k^2/2m+\varepsilon_0$, DOS in 1D, 2D and 3D case is summarized in Tab. 1. At $\varepsilon=\varepsilon_0$, discontinuities (in value or in derivative) occur and they are referred to as the van Hove singularities. They are hallmarks of points where $dE/dk=0$ in the band structure.

Density of states is useful to evaluate quantities that depend on $\vec{k}$ only through energy. To name one of many: number of states with energy less than ε per unit area (i.e. density or concentration of electrons for a given Fermi level $\varepsilon_F=\varepsilon$) can be calculated using Heaviside function $\theta(x)=0/1$ for x negative/non-negative.

$$n=\int\frac{d^3k}{(2\pi)^3}\theta\big(\varepsilon_F-E(\vec{k})\big)=\int_{-\infty}^{\infty}\theta(\varepsilon_F-\varepsilon)g(\varepsilon)d\varepsilon=\int_{-\infty}^{\varepsilon_F}g(\varepsilon)d\varepsilon \tag{4}$$

Corresponding results expressed via Fermi wavevector, $E(k_F)=\varepsilon_F$, are again shown in Tab. 1.

1.3 Coherence length and mean free path

Mean free path λ is related to mobility through $\lambda=\hbar k_F\mu/e$. This quantity, however, has a strictly classical meaning: it is the (mean) distance an electron traverses between two successive scattering events. In order to get rid of $\hbar$, we can write $\lambda=v_F\mu/em$ in terms of the Fermi velocity v_F rather than Fermi wavevector k_F. Mobilities in specially designed systems (see Tab. 2) can

be very high compared to usual silicon-based electronic devices. To get a feeling for the orders of magnitude, consider a rectangular ($L \times w$) MOSFET structure with mobility 100 cm^2/Vs where the thickness of the insulating oxide is a (between the gate and the conducting channel which has the same thickness, let us say). Switching between '0' and '1' states of such transistor corresponds to discharging a capacitor with $C = \epsilon LW/a$ over a resistor of $R = L/(Wa\sigma)$ where $\sigma = ne\mu$ which takes $\sim RC$ (more precisely, voltage on the capacitor decays as $\exp(-t/\tau)$, $\tau = RC$). For $a/L = 0.1$, $\epsilon/\epsilon_0 = 11$ and typical $n = 10^{18}$ cm^{-3}, the operational frequency $f = 1/2\pi RC$ is of the order of GHz.

Now as for the electron wave decoherence, if we blame it on inelastic scattering with phonons, and assume diffusive motion of the electron, the coherence length (independent of dimensionality) can be estimated as $\lambda_\phi \approx \lambda_T = \sqrt{4D\hbar/kT}$. An order-of-magnitude (at best) estimate of the diffusion constant in a clean silver sample is $D = \frac{2}{3}\varepsilon_F\tau/m \approx 1300$ cm^2/s (data from Tab. 1.3 and 2.1 in [1] were used) and the corresponding coherence length $\lambda_\phi \approx 220$ nm at liquid nitrogen temperature. Realistically, coherence length may even exceed 1 μm at low temperatures (see data in Fig. 7 of Ref. [4] or a more recent review of Webb [5]). It should always be $\lambda < \lambda_\phi$. Coherence is a more general notion which also applies to other contexts than transport. For example, coherence time of a qubit is the time over which the full information on the quantum state is preserved. In suitable systems, it can be significantly more than a second even at room temperature [32].

2 Classical, semiclassical, and quantum

The three attributes above are often used without having a sharp boundary between the respective areas in mind. However, thinking about transport, one can roughly get oriented on the following:

- classical transport. Charge carriers (electrons) are treated as charged balls subject to Newtonian mechanics that typically includes electric and magnetic forces. Electrical current between A and B is evaluated by counting the number of particles that have travelled from A to B over a unit of time. This number is multiplied by the (elementary) charge.

- semiclassical transport focuses on the phase space and its occupation described by the distribution function $g(\vec{r}, \vec{k}, t)$. In equilibrium, it equals the Fermi-Dirac distribution and out of equilibrium (e.g. when current-driving electrical field is applied), it is described by the Boltzmann equation. Summing over the occupied states when out from equilibrium yields the current (again after multiplication by change).

  The Boltzmann equation treats electrons *almost as in classical case* but: (a) rather than by momentum $\vec{p}$, the states are identified by their wavevector $\vec{k} = \vec{p}/\hbar$, and (b) the relationship between energy E and $p = |\vec{p}|$ may be more complicated than simply $E = p^2/2m$. *The dispersion relation $E = E(\vec{k})$ is typically taken from a pure quantum-mechanical model e.g. of a crystalline solid.* With some exaggeration: semiclassics – classics = quantum dispersion relation.

- quantum transport requires not only the knowledge of the dispersion relation but also of the corresponding wavefunctions. Without any need to refer to classical notions (particle = charged ball), it calculates current as a response to time-dependent perturbation (current-driving electric fields).

Quite often, all three approaches lead to the same result at the end. A well-known example is the Drude formula

$$\sigma = \frac{n}{m} e^2 \tau \tag{5}$$

which we shall now derive thrice for a 2D electron gas.

2.1 Drude formula (classical)

Among all the assumptions needed here (see [1], Chapter 1, p. 6), let us highlight that electrons thought to be independent from each other, undergo completely randomizing collisions that occur on average after time τ.

In absence of magnetic fields, a particle of charge q emerging from the last collision with random velocity $\vec{v}_0$ at time $t = 0$ will accelerate according to[1] $\vec{F} = m\vec{a}$, i.e. $\vec{v}(t) = \vec{v}_0 + qt\vec{E}/m$. Averaged over random scattering events (that occur around $t = \tau$ on average), the mean particle velocity is $q\tau\vec{E}/m$ and total current (in amperes) from A to B equals $q^2\tau\vec{E}/m$ times the number of particles available and divided by the distance from A to B (we imagine that all particles start at A at the same time and ask how long they need to travel to B). Recast in terms of current density $\vec{j}$ (in Am^{-d+1} for d-dimensional system),

$$\vec{j} = \frac{n}{m} e^2 \tau \vec{E}\,, \tag{6}$$

where n is the charge carrier concentration in m^{-d} (and we inserted $q = -e$ for electrons). Obviously, the conductivity (in inverse $\Omega\mathrm{m}^{d-1}$) of Eq. (5) is recovered for any number of dimensions at once.

[1] Conventions: electric field $\vec{E}$ points from the positive pole to the negative one. Electron charge is $q = -e < 0$ (like in [1]!). Current flows in the same direction as positively charged particles, $\vec{j} = nq\vec{v}$ (opposite to the motion of electrons).

2.2 Drude formula (semiclassical)

To cheat as little as possible, we should start with a Hamiltonian. Consider $\hat{H} = \hat{p}^2/2m^*$ in two dimensions; the corresponding dispersion relation reads $E(\vec{k}) = \hbar^2 k^2/2m^*$ with effective mass m^* and $k = |\vec{k}|$. At this point, we have provided information about the (possibly complicated) band structure that had to be obtained from quantum mechanical calculation. It is the approximation we chose that m^* replacing the vacuum electron mass is the only remnant of this complexity.

Boltzmann equation for the $\vec{k}$-indexed states reads $\partial g/\partial t + (\partial g/\partial t)_{drift} = (\partial g/\partial t)_{scatt}$ — in-flow of particles due to drift or explicit time dependence ('source term', 'adding particles by hand', for instance photoexcitation) equals the out-flow due to scattering. In a steady state (and no externally driven excitation, $\partial g/\partial t = 0$), homogeneous system ($\partial g/\partial x_i = 0$), linear response to $\vec{E}$, and relaxation time approximation (the collision part approximated as shown on the r.h.s. below), this transforms into

$$-\frac{\partial f}{\partial \varepsilon}\vec{v}\cdot q\vec{E} = -\frac{g-g_0}{\tau}\,. \tag{7}$$

Here $g_0(\vec{k}) = f\big(E(\vec{k})\big)$ is the equilibrium (i.e. Fermi-Dirac) distribution and τ is the phenomenological relaxation time. At this point, we introduce the chemical potential μ (appearing in f) and set it for this moment equal to the Fermi level ε_F. It should be noted that in obtaining Eq. (7), (a) the dispersion relation has been employed ($\vec{v} = \hbar^{-1}\partial E/\partial\vec{k}$) together with (b) semiclassical equations of motion $\hbar\dot{\vec{k}} = q\vec{E}$.

The unknown in Eq. (7), g can readily be expressed ($g = g_0 + \tau\cdot\ldots$) and it represents the distribution function (occupation of states) of a system that has been driven out from equilibrium by a weak electric field $\vec{E}$. Current (density) calculated as

$$\vec{j} = \int \frac{d^2k}{(2\pi)^2} q\vec{v}(\vec{k})g(\vec{k}) = q^2\tau\int\frac{d^2k}{(2\pi)^2}\left(-\frac{\partial f}{\partial\varepsilon}\right)\vec{v}(\vec{v}\cdot\vec{E}) \tag{8}$$

turns out to be proportional to $\vec{E}$ times a second-rank tensor whose xx component equals

$$\sigma_{xx} = \frac{e^2}{m^*}\tau\frac{1}{4\pi}\frac{2m}{\hbar^2}\varepsilon_F\,. \tag{9}$$

Making use of the 2D relation $n = k_F^2/(4\pi)$, we once again recover the Drude formula (5). The derivation in 3D goes by analogy, it yields vaguely speaking a *different* result than Eq. (9) (it contains $\varepsilon_F^{3/2}$) which however transforms again into the *same* Drude formula as in (5) if we express it in terms of n rather than ε_F.

Relaxation time can be calculated using the (modified) Fermi golden rule

$$\frac{1}{\tau(\vec{k})} = \frac{2\pi}{\hbar}n_i\int\frac{d^3k'}{(2\pi)^3}|M^{kk'}|^2(1-\cos\theta_{kk'})\delta\big(E(\vec{k})-E(\vec{k'})\big) \tag{10}$$

where $M^{kk'}$ is the matrix element of the single impurity potential between the state before ($\vec{k}$) and after ($\vec{k'}$) scattering and n_i is the density of impurities. The factor $1-\cos\theta_{kk'}$ makes backward scattering more important than forward scattering since the latter should not impede current flow much ($\cos\theta_{kk'} = \vec{k}\cdot\vec{k'}/|\vec{k}||\vec{k'}|$). 'Original' Fermi golden rule is written without this factor and it gives the so called quantum relaxation time as opposed to the transport relaxation time of Eq. (10). The $\vec{k}$–dependence of τ is usually neglected and for some special cases (spherical Fermi surface, isotropic scatterers) is even absent.

For so called uncorrelated short-range scatterers, $V(\vec{r}) = \sum_i V_0\delta(\vec{r}-\vec{r}_i)$ placed at random positions $\{\vec{r}_i\}$, Eq. (10) gives $1/\tau \propto n_i V_0^2 g(\varepsilon_F)$ where $g(\varepsilon_F)$ is the density of states at the Fermi level.

2.3 Drude formula (quantum)

The standard tool to evaluate conductivity quantum-mechanically (QM) is the Kubo formula (see [2] Ch. 3, p. 165) that e.g. determines the linear response of current to time-dependenent electric field $\vec{E}(\vec{q},\omega)$ in terms of a current-current correlator.[2] By carefully taking the static limit $(\omega, \vec{q} \to 0\ )$, we obtain

$$\sigma_{xx}(\varepsilon_F) = \frac{\pi\hbar e^2}{A} \mathrm{Tr}\ \hat{v}_x \hat{A}(\varepsilon_F) \hat{v}_x \hat{A}(\varepsilon_F) \tag{11}$$

where $\hat{A}(z) = \big(\hat{G}^-(z) - \hat{G}^+(z)\big)/(2\pi i)$ is the spectral function (to be imagined as an almost-delta-function of $\varepsilon_F - \hat{H}$) and $\hat{G}^{\pm}(z)$ are the Green's functions $(z - \hat{H} \pm i\varepsilon)^{-1}$. The imaginary part of self-energy, ε corresponds to $\hbar/2\tau$ (spectral broadening[3]) and its real part is disregarded for now. To evaluate the trace in Eq. (11) we choose the basis of eigenstates to $\hat{H}$, i.e. $\langle\psi|\vec{r}\rangle = \exp(i\vec{k}\cdot\vec{r})$, and use the magic formula[4] $\delta(x)\delta(x) \approx (2\pi\varepsilon)^{-1}\delta(x) \equiv (2\tau/h)\delta(x)$. In a 2D system of area A, we get

$$\sigma_{xx} = \frac{e^2}{A}\tau \int \frac{d^2k}{(2\pi)^2} \langle\psi|\hat{v}_x \delta(\varepsilon_F - \hat{H})\hat{v}_x|\psi\rangle = \frac{e^2}{m^*}\frac{\tau}{4\pi}\int_0^\infty d\varepsilon\ \delta(\varepsilon_F - \varepsilon)\frac{2m^*}{\hbar^2}\varepsilon$$

which transforms into Eq. (9) and hence we have again obtained the Drude conductivity (5) of a free 2D electron gas.

Broadening of the spectral function can be evaluated as the imaginary part of the self-energy Σ (rather than ε, it is usually denoted by Γ). From the Dyson equation (see p. 88 and 200 in [2]) $\langle G\rangle = G_0 + G_0\Sigma\langle G\rangle$ for clean-system Green's function G_0 and dirty-system $\langle G\rangle$ averaged over impurity ensemble configurations, it is possible to express

$$\Gamma \equiv \mathrm{Im}\ \Sigma = \langle V\, \mathrm{Im}\ G_0 V\rangle = n_i V_0^2 g(\varepsilon_F), \tag{12}$$

where the last step holds for uncorrelated short-range scatterers. The brackets $\langle\cdot\rangle$ indicate averaging over impurity configurations, relaxation time equals $\hbar/2\Gamma$.

3 Magnetoresistance

In zero magnetic field, resistivity tensor ρ is diagonal. That is to say, electric field $\vec{E}$ that arises when we create current density $\vec{j}$, will be parallel to $\vec{j}$ since $\vec{E} = \rho\vec{j}$. When magnetic field is switched on, transversal electric field (E_H) appears due to the classical Hall effect hence $\vec{j}$ and $\vec{E}$ are no longer parallel and ρ must attain off-diagonal components. We now derive these classicaly.

In a very long and narrow stripe (let its width be w), when steady state is reached, current will be flowing along x with drift velocity v_d ($j = -nev_d$). This is only possible when Lorentz force is fully compensated by E_H — its microscopical origin is the following: before steady state was reached, electrons were deflected towards one of the edges of the stripe; charge inbalance arose and hence E_H became nonzero. We obtain $\vec{E}_H = -\vec{v}_d \times \vec{B} = \vec{j} \times \vec{B}/ne$. In terms of the voltage U_{xy} measured accros the stripe, $U_{xy}/w = IB/(new)$ which implies Hall resistance $R_H = B/ne$ (in 2D) or $R_H = B/net$ (in a 3D stripe of thickness t). Longitudinal voltage will not be influenced by B (Lorentz force has been compensated) and we may write

$$\vec{E} = \begin{pmatrix} \rho_0 & B/ne \\ -B/ne & \rho_0 \end{pmatrix} \vec{j}. \tag{13}$$

[2] One of the currents has a clear origin: we are interested in current response. The other current comes from the term $\vec{A}\cdot\vec{j}$ that describes the force exerted by $\vec{E} = -\partial\vec{A}/\partial t$ on electric charge; this $\vec{A}\cdot\vec{j}$ corresponds to $\vec{E}\cdot\vec{v}$ in Eq. (7).

[3] To see this, plot the (expectation value of) spectral function $\hat{A}$ as a function of z for a system with single energy level at ε_0. It is a Lorentzian with FWHM $2\varepsilon = \hbar/\tau$. Beware of the slightly confusing notation: ε is the broadening, not to be confused with the line position ε_0, Fermi level ε_F or the Fermi-Dirac function variable in $df/d\varepsilon$.

[4] Reasoning behind is based on taking $(\varepsilon/\pi)\cdot(x^2+\varepsilon^2)^{-1} = \delta_\varepsilon(x)$ as an approximate expression for Dirac delta distribution ($\lim \delta_\varepsilon = \delta$). Just as it holds $\int \delta_\varepsilon(x)dx = 1$, we then require $\int c[\delta_\varepsilon(x)]^2 = 1$ and find c (which however depends on ε), that makes $\lim\ c[\delta_\varepsilon(x)]^2 = \delta(x)$.

Conductivity tensor σ is simply the inverse of ρ and it has the same form as what is implied by Chamber's solution[3] to the Boltzmann equation.[5] Details of this calculations are given in Ref. [1] (p. 248), we only sketch the main steps here. When evaluating $g(\vec{k},t)$, we follow trajectory of the particle that will occupy the state $\vec{k}$ in time t. This non-equilibrium and equilibrium occupation numbers will be denoted by $g(t)$ and $g_0(t)$. It holds

$$g(t) = g_0(t) - \int_{-\infty}^{t} dt' P(t,t') \frac{dg_0(t')}{dt'} \tag{14}$$

where $P(t,t')$ is the probability that the particle won't suffer any scattering between t and t' (and $t > t'$). In the RTA, this probability is $\exp(-(t-t')/\tau)$. We then insert the drift and diffusion terms of the Boltzmann equation for $dg_0(t')/dt'$, i.e. the same terms that produced the l.h.s. of Eq. (7) and finally obtain

$$g(\vec{k},t) = g_0(\vec{k}) + \int_{-\infty}^{t} e^{-(t-t')/\tau} \left(-\frac{f_0}{\varepsilon}\right) \vec{v}(\vec{k}(t')) \cdot (-e\vec{E})\,. \tag{15}$$

This is the non-equilibrium distribution function that can be inserted into formula for current (8) and the proportionality constants between components of $\vec{j}$ and components of $\vec{E}$ are the conductivity tensor components.

At $B = 0$, the integral in Eq. (15) simply produces a factor of τ because $\vec{k}$ does not vary much in time, it only slightly accelerates between $\vec{k}$ and $\vec{k} - e\vec{E}\tau$ between two scattering events. We are now in the diffusive regime (as opposed to ballistic regime where scattering events are rare), $eE\tau \ll k_F$, hence we drop the t' dependence as announced in the footnote before Eq. (14).

On the other hand, for large magnetic fields (the precise condition will be revealed soon), electron's $\vec{k}$ will change rapidly in time albeit *not* in its magnitude. Classically, this corresponds to the statement that homogeneous magnetic field cannot change particle's energy (Lorentz force is perpendicular to velocity) and we will now show how to describe the situation semiclassically in the language needed for Eq. (15).

What lies at the heart of semiclassical dynamics are naturally equations of motion:

$$\dot{\vec{r}} \equiv \vec{v} = \frac{1}{\hbar}\nabla_k E(\vec{k})\,, \qquad\qquad \hbar\dot{\vec{k}} = q(\vec{E} + \vec{v} \times \vec{B})\,. \tag{16}$$

Insert the first equation into the second one and what you then have to face is a system of two differential equations for $k_x(t)$ and $k_y(t)$ provided that we (for the sake of simplicity) put $k_z(t) \equiv 0$. For zero electric field, these equations can be readily solved and yield $(k_x, k_y) = k_0(\cos\omega t, \sin\omega t)$ where $\omega = eB/m$ and k_0 is determined by the initial condition. Being interested in states that determine system's transport properties, we will take $k_0 = k_F$.

This solution of Eq. (16), inserted into Eq. (15) and having evaluated the time integral, the non-equilibrium distribution plugged into Eq. (8) produces the current in linear response to electric field (and under arbitrary B). If we carried out the calculation for electric field along x, we will nevertheless find nonzero current along both x and y hence the conductivity tensor will be non-diagonal. In particular, $\sigma_{xx} = j_x/E_x$ and $\sigma_{xy} = j_x/E_y$. The whole tensor assumes the form

$$\sigma = \frac{\sigma_0}{1 + (\omega\tau)^2} \begin{pmatrix} 1 & -\omega\tau \\ \omega\tau & 1 \end{pmatrix} \tag{17}$$

where σ_0 is the Drude conductivity. As promised, the inverse of σ in Eq. (17) exactly equals the classical result for resistivity in Eq. (13).

Experiments confirm the behaviour of Eqs. (13,17) in some range of parameters, but under more "extreme conditions" significant deviations occur as shown in Fig. 1: longitudinal resistance (ρ_{xx}) starts to oscillate and transversal resistance (ρ_{xy}) develops plateaus close to quantized values h/e^2 times one over integer number (most prominently even number). The whole range of these effects is covered by the so-called Shubnikov–de Haas oscillations and the quantum Hall effect.

[5] Boltzmann equation in the form of Eq. (7) already contains certain assumptions which are violated when magnetic field is present. In illustrative terms, without magnetic field, electron states accelerated by electric field do not "move much around" in the phase space.

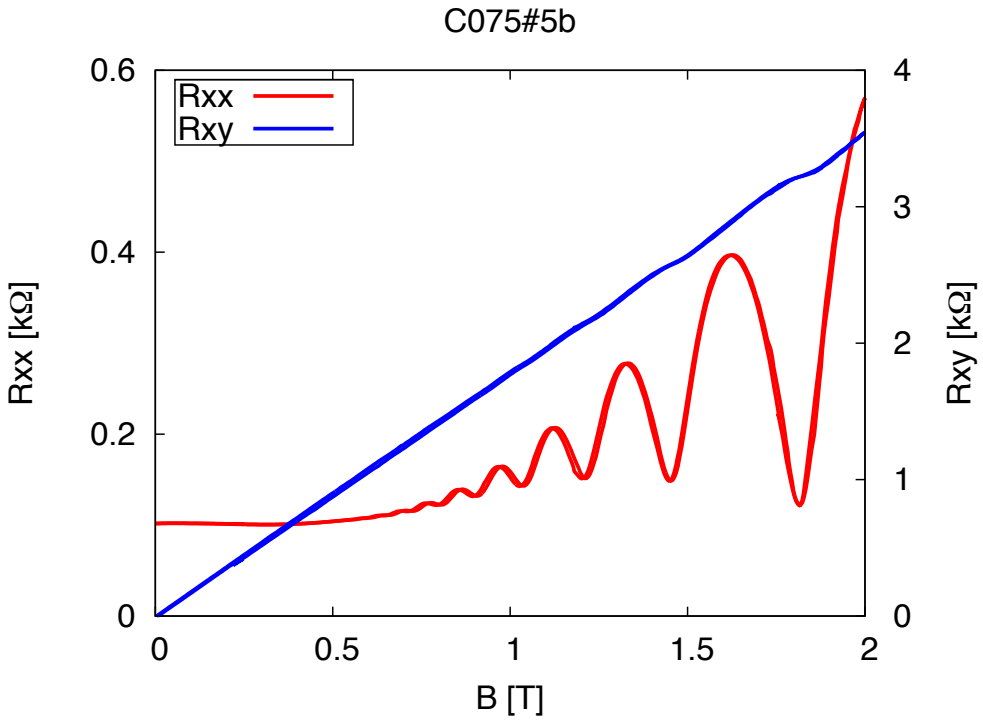

Figure 1: Shubnikov-de Haas oscillations in a GaAs/AlGaAs Hall bar containing 2DEG. (Measured by L. Nádvorník.)

3.1 Shubnikov–de Haas oscillations

Oscillations shown in Fig. 1 originate from quantum mechanics and cannot be explained by semiclassical theory of charge carriers as it is. However, we can borrow the substantial quantum feature of electrons, described by their wavefunction, and explain what corrections to Eq. (17) should be expected.

What quantum mechanics tells us (and what semiclassical theories do not know[6]) is that the wavefunction of an electron must unambiguous that is, if an electron follows a cyclotron trajectory, its phase must be the same modulo 2π at the beginning and at the end of each period. Alternatively, one could require the length of the closed trajectory to be an integer multiple of electron's wavelength. More precisely, this Bohr-Sommerfeld quantization condition reads

$$\oint \vec{k} \cdot d\vec{r} = 2\pi(n+\gamma) \tag{18}$$

where n is an integer and γ a quantum-mechanical correction (related to the fact that for instance Landau level energies in units of $\hbar\omega$ are $n+\frac{1}{2}$ and not just n; see Sec. 6.3). Inserting $\hbar\vec{k} = m\vec{v} - e\vec{A}$ into the l.h.s. of Eq. (18), we get $2\pi R_C^2 eB/\hbar - (e/\hbar)\oint \vec{A}\cdot d\vec{r} = 2SB/(\hbar/e) - SB/(\hbar/e) = 2\pi SB/(h/e)$ in terms of magnetic flux $\Phi = SB$ passing through electron's cyclotron orbit. Except for the corrective term γ, Eq. (18) can be reexpressed as a requirement that Φ be an integer multiple of magnetic flux quantum $\Phi_0 = h/e$.

Let us apply condition (18) to states at the Fermi level. As we could have verified when solving Eqs. (16), real-space trajectories (cyclotron orbits) correspond to a periodic motion in the $\vec{k}$-space via a mapping that consists of a 90° rotation and rescaling by[7] $m\omega/\hbar$. A 2D electron will orbit around the Fermi surface in the $\vec{k}$-space and so will a 3D electron which moreover never leaves

[6] There is not a universal rule of what one should call "semiclassical theory". The construction we are just about to described is sometimes also called "semiclassical theory".

[7] That is: take $\vec{r}(t)$, multiply it by $m\omega/\hbar$, rotate it and you get $\vec{k}(t)$.

one single plane in the $\vec{k}$-space that is perpendicular to $\vec{B}$ (3D electron thus remains confined to a section of the Fermi surface in the direction $\perp \vec{B}$). Owing to the rescaling, Eq. (18) can be rewritten as

$$\frac{1}{B} = \frac{2\pi e}{A\hbar}(n + \gamma) \tag{19}$$

where A is the area of the Fermi surface (section). Whenever this condition is violated, the state at the Fermi energy is quantum-mechanically forbidden. In 2D, the density of states (DOS) at such Fermi energy (or equivalently, carrier concentration) should be zero, in 3D the corresponding Fermi-surface section does not contribute to the DOS. Lowered DOS will make the conductivity drop, as demonstrated in Eq. 21, and oscillations in ρ_{xx} arise which are periodic in $1/B$. If we measure their period in $1/B$, the area of the Fermi surface may be inferred and hence also the total density of carriers.

3.2 Selected other types of magnetoresistance

Except for SdH oscillations, where can other deviations from Eq. (13) come from? Weak localization is also an example of a magnetoresistive effect. There is however a whole myriad of other reasons why resistance may depend on magnetic field.

3.2.1 Geometrical magnetoresistance

A very common reason for the resistance to depend on magnetic field is that the current lines bend within a specific conductor geometry, depending on B. To find out how we have to find electrostatic potential in the sample by solving Laplace equation $\Delta V = 0$ and combine it with $\vec{j} = \sigma\nabla V$ where σ is given by Eq. (17). Here, boundary conditions play a crucial role: they may be either $\vec{n} \cdot \vec{j} = 0$ (insulating surface defined by its normal vector $\vec{n}$), V equal to constant (equipotential surface, i.e. an ideal metallic contact) or possibly more complicated.

Two good examples which can be solved without involved numerical calculations are a very narrow conducting channel (such as a Hall bar) and a Corbino disc (area between two concentric rings which represent contacts) or its non-periodic modification (infinite slab). Concisely summarized, the current lines will be mostly parallel to the Hall bar while equipotentials will be parallel to the slab. Because of $\vec{j} = \sigma\nabla V$, the two (equipotentials and current lines) will not be perpendicular, the angle between $\vec{E}$ and $\vec{j}$ being called the Hall angle. Resistance of such a device (voltage drop between contacts divided by current) turns out to be independent from B for the Hall bar but it equals $(1 + \omega^2\tau^2)l/w\sigma_0$ for a 2D slab of width w and thickness l. Several more complicated situations are shown in Fig. 2.

3.2.2 Multi-band magnetoresistance

So far, we only considered systems with a single band of charge carriers described by dispersion $E(\vec{k})$. Conductivity in such a system given by Eq. (17) inverted into resistivity yields constant (B-independent) ρ_{xx}. If more bands cross the Fermi level then individual conductivities simply add but the miraculous single-band property of ρ_{xx} is lost and magnetoresistance arises.

It is an instructive exercise to calculate ρ_{xx} for two bands with equal carrier concentrations $n_1 = n_2$ and different mobilities $\mu_1 > \mu_2$. Using electron mobility that fulfils $\sigma_0 = ne\mu$, Eq. (17) can be rewritten as

$$\sigma_1 = \frac{n_1 e\mu_1}{1 + (\mu_1 B)^2} \begin{pmatrix} 1 & -\mu_1 B \\ \mu_1 B & 1 \end{pmatrix} \tag{20}$$

for the first band and an analogous expression describes σ_2. In the limit $\mu_1 B \ll 1$, the diagonal components of the inverse of $\sigma_1 + \sigma_2$ acquire a positive correction to $1/(n_1 e\mu_1 + n_2 e\mu_2)$ which is proportional to B^2. [6]

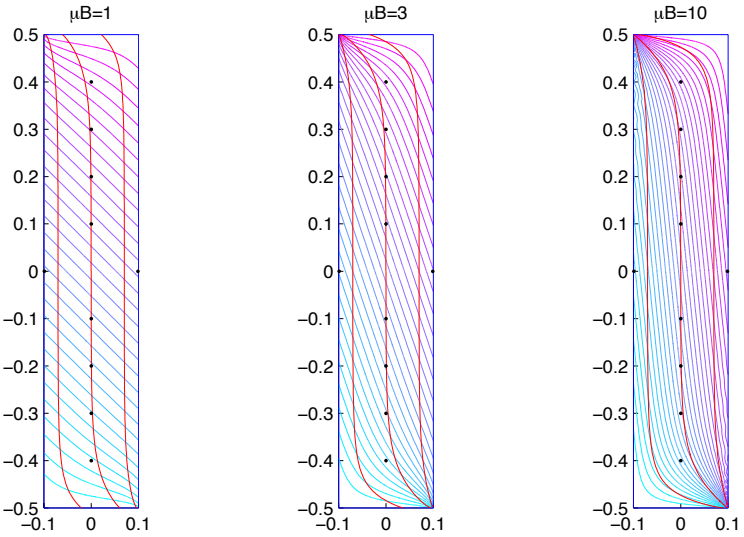

Figure 2: Realistic current profiles (red lines) and equipotentials (dense blue/magenta lines) in a finite Hall bar. (note that $\mu B = \omega\tau$) Courtesy of V. Novák.

3.2.3 Superlattices

A large class of magnetoresistive phenomena can appear due to broken translational symmetry of the system. Apart from the already mentioned geometrical magnetoresistance (i.e. cutting a finite piece from an infinite 2DEG) this can happen by imposing a periodic external potential on a ~ 10 to 100 nm scale (superlattice) as it can be achieved by various methods, most prominently by optical or electron lithography. Possible effects in a 2DEG include

- Weiss oscillations that occur under 1D modulation [8]. They result from commensurability of the classical cyclotron radius (of electrons at the Fermi level) with the potential period a;
- Hofstadter butterfly spectrum in 2D superlattices manifesting[8] itself in non-monotonous ρ_{xy} [9]. Here the commensurability occurs between a and the magnetic length $\ell_0 = \sqrt{\hbar/eB}$;
- magnetic breakdown which is a term used to describe oscillatory ρ_{xx} appearing even in situations where electrons do not follow closed orbits (and hence the Bohr-Sommerfeld quantization cannot be applied as it was in a translationally invariant 2DEG). [13] Strongly modulated 2DEG is a simple model system where this effect can be studied. [12]

3.2.4 Quantum oscillations

To illustrate the connection between conductivity and DOS at the Fermi level, we recall the semiclassical $B = 0$ result

$$\sigma_{xx} = \tau e^2 \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} v^2(\vec{k}) \delta(\varepsilon_F - E(\vec{k})) = \tfrac{1}{2}\tau e^2 v_F^2 g(\varepsilon_F) \tag{21}$$

which is equal to $e^2\tau n/m$. Here, the DOS is a smooth function of ε_F, cf. Tab. 1, so there seems to be no way open to obtain oscillations in conductivity.

The same result as in Eq. (21) can also be derived from the Kubo formula quantum-mechanically [see Eq. (11) and its subsequent evaluation]. It can therefore be expected that if we perform the full QM calculation including magnetic field which yields $g(\varepsilon_F)$ oscillating as a function of B, we

[8] Note however, that the non-monotonicity alone is not an unambiguous evidence here. Complicated FQHE (competition between different incompressible states [10]) or an integer QHE in a multi-band system may also produce $d\rho_{xy}/dB$ of varying sign. [11]

recover SdH oscillations and this expectation proves correct [14]. Two factors that can suppress these oscillations should be mentioned: disorder and temperature. By effectively smearing out the oscillatory features of the DOS, the SdH oscillations disappear when $\omega\tau \gtrsim 1$ or $kT \gtrsim \hbar\omega$.

The extreme quantum limit of the SdH oscillations is the quantum Hall effect.

3.2.5 Resistance of magnetic materials

Concisely summarized, efficiency of a system in transporting charge (i.e. conductivity) depends on three factors: number of states that participate in transport, their group velocity, and scattering. Any of these can be influenced by magnetic field and magnetoresistance ensues. SdH oscillations are an example of the first cause (modulation of DOS), classical Hall effect relates to the second.[9]

The easiest way to influence scattering by magnetic field is to focus on magnetic systems. The basic idea behind (one type of) magnetoresistance is that the magnetic moments on which electrons may scatter, are somehow disordered (when temperature is larger than zero and smaller than Curie temperature) and this disorder can be suppressed by external magnetic field $\vec{B}$ which helps them align along $\vec{B}$. Scattering is then reduced. A nice model system where this effect was observed is the dilute magnetic semiconductor[10] (Ga,Mn)As, see for example Fig. 4 in Ref. [15]. Additionally, this magnetoresistance is anisotropic with respect to crystallographic directions and thus resistance may be varied even by keeping the magnitude of $\vec{B}$ constant and only rotating it in the sample [16].

Both effects (magnetoresistance and anisotropic magnetoresistance) are frequent in most of magnetic materials (iron, cobalt, permalloy, ...)

3.3 Comment on measuring the Fermi surfaces

In a 3D system, the gaps between Landau levels close due to continuous part of the spectrum corresponding to electron's motion parallel to the applied magnetic field. No QHE can be expected but oscillation in the DOS are preserved. From their period, the area of the Fermi surface cross section can be determined and by applying mangetic field in various directions, the Fermi surface can often be completely reconstructed.

See Chapter 14 in Ashcroft&Mermin [1].

3.4 Quantum Hall effects

Quantum Hall effects (QHE) are explained in PS's notes. Here, only some additional remarks follow.

When Fermi level lies in a gap (where it is pinned to localized states), we find ourselves in an unusual situation where both (longitudinal) resistivity and conductivity are zero. Indeed,

$$\sigma = \begin{pmatrix} 0 & -ne/B \\ ne/B & 0 \end{pmatrix} \qquad \rho = \begin{pmatrix} 0 & B/ne \\ -B/ne & 0 \end{pmatrix} \tag{22}$$

Theoretically, the zero on diagonal of σ can be understood based on Eq. (21) and the off-diagonal term using Kubo-Středa formula [7]

$$\sigma_{xy}(\varepsilon_F) = \frac{i\hbar e^2}{2}\mathrm{Tr}\,\left[v_x G^+ v_y A - v_x A v_y G^-\right] + e\frac{\partial n}{\partial B}\bigg|_{E=\varepsilon_F}$$

where $\partial n/\partial B = \nu e/h$ as long as the Fermi level does not leave the gap between Landau levels (and because the spectral functions A make the other term vanish, we are left with σ_{xy} quantized to integer multiples of e^2/h). This brings us to the issue of the origin of the gap: zero resistivity

[9] This analysis should not be taken too strictly, some effects may not fit in any of the classes (e.g. quantum interference such as the weak localization).

[10] This material can be viewed as the usual GaAs with a small part of gallium substituted by manganese atoms which carry magnetic moment.

observed in experiments at integer values of ν can be attributed to the energy gap between LLs which are found in the spectrum of a single electron confined to a plane and subject to magnetic field. However, where can possibly the gap come from at fractional fillings?

The answer (proposed probably by Robert B. Laughlin) is — from electron-electron interactions. Full many-body Hamiltonian of N_e electrons in magnetic field reads

$$H = \frac{1}{2m}\sum_{i=1}^{N_e}\left(\vec{p}_i - q\vec{A}(\vec{r}_i)\right)^2 + \frac{e^2}{4\pi\varepsilon}\sum_{i<j}\frac{1}{|\vec{r}_i - \vec{r}_j|} \tag{23}$$

where ε is the material permittivity (e.g. $\approx 12.7\varepsilon_0$ in GaAs). While the second term makes any hope for exact solution of Schrödinger equation almost equal to zero (but not quite), we will argue below that it is indeed responsible for the occurence of a gapped ground state at fractional filling factor $\nu = 1/3$.

Before we do that, let us first consider scaling of eigenvalues of this H with magnetic field. Assume constant[11] N_e and consider fixed filling factor ν. While the first term will be still varying $\propto B$ (just as individual LLs), the second term will be proportional to $\sqrt{n}$ and that is in turn $\propto \ell_0 \propto \sqrt{B}$. We can therefore expect that in very strong magnetic fields the basic structure of spectrum will be composed of Landau levels whose macroscopic degeneracy will, however, be lifted. On the other hand, if the first term of (23) is not much larger than the interactions (which occurs for weaker magnetic fields) LLs will completely disappear and we obtain a complicated interaction-dominated spectrum. This situation is called (strong) "LL mixing".

Reasons that (probably) led Laughlin to guessing his WF are summarized in Ref. [17] (see p. 31 of the electronic version).

Quasiparticles with fractional charge: experimental evidence of current being carried by particles with charge $e/3$ is mentioned in slides shown in the lecture.

A good book covering large part of the FQHE is Ref. [18].

3.4.1 Quantum Hall ferromagnets

Consider the situation of integer ν where there are two (nearly) degenerate LLs and below them $\nu - 1$ completely occupied LLs. The two degenerate LLs then offer $2eB/h$ single-particle states to fill with only eB/h electrons (per unit area A) at hand and without taking the electron-electron interaction into account, this can be done by very many equally good ways. 'Equally good' means that all of the $(2N!)/(N!)^2$ ways of distributing the $N = AeB/h$ electrons have the same (minimum) energy.

Electron-electron interaction changes the situation dramatically. In order to maximise the exchange energy gain, electrons will prefer to have their spins mutually aligned. Also, their *pseudospin*

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \tag{24}$$

will be aligned and depending on the nature of the two crossing LLs, an anisotropy energy will be associated with the choice of θ (states with $\theta = 0$ and $\theta = \pi$ correspond to an electron in the one or in the other LL, respectively). Within Hartree-Fock approximation and assuming the absence of LL mixing, this energy is

$$E(\theta) = (-b + U_{\rho,\sigma})\cos\theta + \frac{1}{2}U_{\sigma,\sigma}\cos^2\theta \tag{25}$$

as given by Eq. (3) in Ref. [31]. Splitting of the nearly degenerate LLs is $2b$ and $U_{\rho,\sigma}, U_{\sigma,\sigma}$ describe the electron-electron interaction. For $\nu = 2$, it holds[12] $U_{\sigma,\sigma} = -\frac{1}{16}\int q^3 \exp(-q^2/2)d^2q < 0$ and hence if the first term in Eq. (25) is annulated by tuning b, this is an Ising-type QHF.

[11] This corresponds to evaluating energy per particle.

[12] $U_{\rho,\sigma}, U_{\sigma,\sigma}$ are evaluated from Eq. (4) and Tab. I in Ref. [31]. The appropriate Laguerre polynomials in Tab. I are $L_0(x) = 1$ and $L_1(x) = -x + 1$.

4 Superconductivity

This note should be taken as a context for the discussion of Josephson effects rather than a thorough introduction to superconductivity. We will largely follow Ref. [20].

Low-temperature (or type-I) superconductivity arises because of the formation of a gapped ground state consisting of Cooper pairs which are difficult to break and hence scattering is suppressed. Roughly speaking, zero resistance state persists up to temperatures $kT \sim \Delta_0$ and the gap Δ_0 is the energy needed to break a Cooper pair. The ground state is described by the BCS theory of individual Cooper pairs (by L.N. Cooper) that condense into a state with macroscopically many of them (proposed by J.R. Schrieffer) by virtue of an effective electron-electron interaction due to phonons (studied by J. Bardeen and D. Pines).

4.1 Cooper pairs

Consider Fermi sea of non-interacting electrons (e.g. Fermi sphere for electrons obeying $E(\vec{k}) = \hbar^2 k^2/2m$). Will there be some significant change if electron-electron interactions are switched on? L.N. Cooper found[21] that the answer is positive under specific circumstances and the system can lower its energy by forming a pair of electrons outside the original Fermi surface.

Wavefunction of such a pair will be $|\mathrm{FS}\rangle \otimes |\psi\rangle$ (Fermi sea plus the pair) and $\langle \vec{r}_1, \vec{r}_2|\psi\rangle = \psi(\vec{r}_1, \vec{r}_2) = \varphi_{\vec{q}}(\vec{r}) e^{i\vec{q}\cdot\vec{R}}$ with $\vec{r}_{1,2}$ denoting the coordinates of the two electrons in the pair and $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$, $\vec{r} = \vec{r}_1 - \vec{r}_2$. When no current is flowing, $\vec{q} = 0$, the wavefunction of the pair is a superposition of states with opposite wavevectors (and opposite spin [23])

$$\psi(\vec{r}_1, \vec{r}_2) = \varphi_{q=0}(\vec{r}) = \sum_{\vec{k}} a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} = \sum_{\vec{k}} a_{\vec{k}} \underbrace{e^{i\vec{k}\cdot\vec{r}_1} e^{-i\vec{k}\cdot\vec{r}_2}}_{|\vec{k},-\vec{k}\rangle} \tag{26}$$

whose coefficients can be found by solving Schrödinger equation. If we measure all energies from the Fermi level ε_F, it reads $(\hat{H}_0 + \hat{V})|\psi\rangle = E|\psi\rangle$ where $\hat{H}_0|e^{i\vec{k}\cdot\vec{r}_1}\rangle = \varepsilon_{\vec{k}}|e^{i\vec{k}\cdot\vec{r}_1}\rangle$, $\varepsilon_{\vec{k}} = E(\vec{k}) - \varepsilon_F$ is the single-particle part of the Hamiltonian and $\hat{V}$ is the (effective) electron-electron interaction. Inserting Eq. (26) into the Schrödinger equation and comparing the coefficients in front of $|\vec{k}, -\vec{k}\rangle$, we get

$$2\varepsilon_k a_k + \sum_{k'} V_{kk'} a_{k'} = E a_k \tag{27}$$

where we have abbreviated $\langle \vec{k}, -\vec{k}|V|\vec{k}', -\vec{k}'\rangle$ as $V_{kk'}$ (we will omit the vector signs henceforth for brevity). Next, we consider a model interaction $V_{kk'} = \lambda w_k w_{k'}$ of strength λ, which allows scattering of the pairs only outside the Fermi sea ($w_k = 0$ when $\varepsilon_k < 0$) and later, we will also make use of the simplifying assumption $w_k = 0$ for $\varepsilon_k > \hbar\omega_0$. Eq. (27) can then be rewritten as $(E - 2\varepsilon_k)a_k = \lambda w_k \sum_{k'} w_{k'} a_{k'}$, or using

$$C \equiv \sum_{k'} w_{k'} a_{k'} \quad \text{also as} \quad \frac{\lambda w_k C}{E - 2\varepsilon_k} = a_k. \tag{28}$$

The second equation in (28) can then be re-inserted into the first one and cancelling C on both sides, we arrive at

$$\frac{1}{\lambda} = \sum_k \frac{w_k^2}{E - 2\varepsilon_k}. \tag{29}$$

Here, we got rid of a_k, hence we cannot use this equation to determine the wavefunction in Eq. (26) but we still can use it to find its energy E. Graphical solution of this equation is shown in Fig. 3. For arbitrarily weak attractive interaction between electrons ($\lambda < 0$), there is always a bound state (solution with $E < 0$). This can be the case when electron-phonon interactions are stronger than the usual electron-electron repulsion.[22]

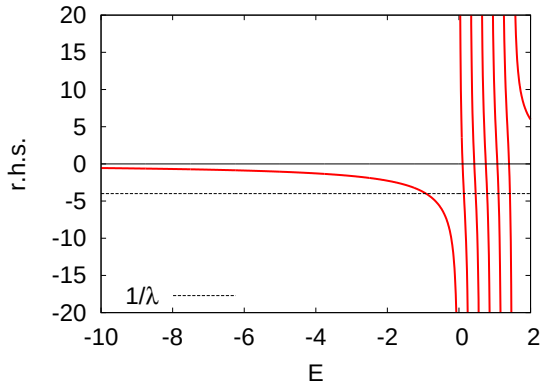

Figure 3: Graphical solution of Eq. (29) for negative λ. Wavevectors in the sum are taken to be discrete but the conclusion about a bound state (solution with $E < 0$) remains true even in the limit where k becomes continuous.

4.2 BCS wavefunction

The existence of a state with $E < 0$ suggests that not only one but more Cooper pairs will form is the effective attractive interaction is present (the Fermi sea is unstable). A many-body wavefunction can be written for such a state [23]

$$|\mathrm{BCS}\rangle = \prod_k (u_k + v_k c_k^+ c_{-k}^+)|\mathrm{vac}\rangle \tag{30}$$

in terms of Cooper-pair creation operators $c_k^+ c_{-k}^+$ and certain numerical coefficients u_k, v_k. These coefficients can be expressed (see below) in terms of Δ_0 which is in equilibrium just the square root of Cooper pair density ρ. When current is flowing, Δ_0 becomes complex, $\Delta_0 = \sqrt{\rho} e^{i\phi(r)}$, and the macroscopically coherent position-dependent phase $\phi(r)$ gives rise to the Josephson effects.

Coefficients u_k, v_k in the BCS wavefunction are normalized to $|u_k|^2 + |v_k|^2 = 1$ and if we take $v_k = 1$ for $\varepsilon_k < 0$ and $v_k = 0$ for $\varepsilon_k > 0$, Eq. (30) recovers the (normal) Fermi sea state. The relevant states are added to vacuum $|\mathrm{vac}\rangle$ pair by pair but otherwise, Eq. (30) is then just an alternative way of writing $\Pi_k c_k^+ |\mathrm{vac}\rangle$ where the product goes over all states with energy less than E_F ($\varepsilon_k < 0$). Situation is different when the electron-phonon interaction is switched on. By minimising the energy implied by Hamiltonian $\hat{H}_0 + \hat{V}$, see text around Eq. (27), we arrive at

$$u_k = \frac{E_k + \varepsilon_k}{\sqrt{\Delta_0^2 + (E_k + \varepsilon_k)^2}}, \qquad v_k = \frac{\Delta_0}{\sqrt{\Delta_0^2 + (E_k + \varepsilon_k)^2}} \tag{31}$$

for $|\varepsilon_k| < \hbar\omega_D$ while for all other values of k, the values of v_k remain 0 or 1 as in the Fermi sea state. The BCS wavefunction with u_k, v_k of this form is fundamentally different from the normal Fermi sea, for example it does not have a sharp number of particles, and it exhibits an off-diagonal long-range order.

Energy of the BCS wavefunction is lower than that of the Fermi sea by $\frac{1}{2} N_0 \Delta_0^2$ where N_0 is the density of states at the Fermi level of the normal state. Parameter Δ_0 in Eq. (31) is found in the energy minimisation procedure, $\Delta_0 = \hbar\omega_D / \sinh(1/N_0|\lambda|)$ and $E_k^2 = \varepsilon_k^2 + \Delta_0^2$.

5 Spin and transport

. . . a very broad topic indeed, only selected effects are discussed.

5.1 Hanle effect

An analogy of magnetic-field-induced spin precession of a free electron in vacuum can be also performed in solid state. Such experiments with non-local spin valves were first performed in van Wees group [33]. The signal is proportional to the spin polarisation that arrives at the second ('read-out') electrode by diffusion characterised by D. Probability to travel a distance of L in such a manner is $P(t) = \exp(-L^2/4Dt)/\sqrt{4\pi Dt}$ where t is the time of travel. At the same time, resulting spin (its z-component) also depends on t as $\cos\Omega_L t$ where $\Omega_L = gBe/2m$ is the Larmor frequency. The total signal is then [33]

$$V \propto \int dt\ P(t) e^{-t/\tau_{sf}} \cos\Omega_L t \tag{32}$$

where we also account for the possibility of a random spin flip (spin relaxation time τ_{sf}).

6 Miscellanea

6.1 Shot noise

Consider a time-dependent random quantity $A(t)$ and let us define the spectral density of its fluctuations

$$S_A(\omega) = \lim_{T\to\infty} \frac{\langle|\tilde{A}_T(\omega;t_0)|^2\rangle}{T} \text{ where } \tilde{A}_T(\omega;t_0) = \int_{t_0}^{t_0+T} dt\ e^{i\omega t} A(t). \tag{33}$$

We will show that when the quantity in question is electric current (carried by individual electrons traversing the measurement point at times t_n),

$$A(t) = a\sum_n \delta(t-t_n), \tag{34}$$

i.e. the amplitude of each peak $a = e$ (charge transported by one electron), the spectral density of current fluctuations is independent of ω at finite frequencies and in particular

$$S_I(\omega \neq 0) = e\bar{I} \tag{35}$$

where $\bar{I} = \nu e$ is the average current (ν is the average number of t_n's per unit time).

To show that Eq. (35) holds, we use the Wiener–Khinchin theorem (p. 214 in [19])

$$2\pi\delta(\omega-\omega')S_A(\omega) = \langle\tilde{A}(\omega)\tilde{A}^*(\omega')\rangle \tag{36}$$

where $\langle\ldots\rangle$ denotes averaging over the randomly distributed $\{t_n\}$. On the left-hand side of Eq. (36), we have taken the $T\to\infty$ limit to $\tilde{A}_T(\omega;t_0)$ and assume translational invariance in time (independence on the starting time of the measurement t_0). The Fourier transform of $A(t)$ is

$$\tilde{A}(\omega) = a\tilde{f}(\omega)\sum_n e^{i\omega t_n}$$

and for $f(t) = \delta(t)$, it is $\tilde{f}(\omega) = 1$. We proceed by evaluating the right-hand side of Eq. (36)

$$2\pi\delta(\omega-\omega')S_A(\omega) = \left\langle a^2 \sum_{n,n'} e^{i\omega t_n} e^{-i\omega' t_{n'}} \right\rangle = a^2\left[\left\langle\sum_n e^{i(\omega-\omega')t_n}\right\rangle + \left\langle\sum_n e^{i\omega t_n}\right\rangle\left\langle\sum_n e^{-i\omega' t_n}\right\rangle\right]$$

where we split the $n = n'$ (first term) from the $n \neq n'$ cases (second term). Using $\langle\sum_n \exp(i\omega t_n)\rangle = \int\exp(i\omega t)\nu dt = 2\pi\nu\delta(\omega)$, we can now write

$$2\pi\delta(\omega-\omega')S_A(\omega) = 2\pi\nu a^2\delta(\omega-\omega') + (2\pi\nu)^2 a^2\delta(\omega)\delta(\omega'). \tag{37}$$

Let's focus on $\omega' \neq 0$. The second term in Eq. (37) then drops out and we obtain $S_A(\omega) = \nu a^2$ for $\omega \neq 0$. When A is electrical current, i.e. $a = e$ in Eq. (34), we immediately recover Eq. (35). Current fluctuations in the (classical) shot-noise regime are proportional to the average current and the proportionality constant is the charge of individual carrier. If electrical charge were not quantized, there would be no shot noise.

6.2 Metal-insulator transitions

Introductory paragraph of a relatively recent article of Aleiner et al. (PRL 114, 076802) illustrates pretty well the context: *Quantum coherence of electron motion dramatically affects low temperature phenomena in disordered conductors. Anderson localization is the most profound of them, but even in the metallic regime, where quantum effects are relatively small, they give rise to a number of dramatic effects due to their extreme sensitivity to magnetic field and inelastic processes. Celebrated examples are universal conductance fluctuations (UCF), magnetoresistance in weak magnetic fields, and Aharonov-Bohm (AB) oscillations in thin mesoscopic cylinders and rings.*

Anderson-type metal-insulator transition (MIT) [34] be explained in the following way. Upon doping, the $T \to 0$ conductivity of a 3D semiconductor remains zero up to a certain critical dopant concentration n_c which can be linked — at least naively — to the effective Bohr radius $a_{B,eff}$ by $a_{B,eff}^3 n_c \approx 0.053$ (Eq. 18.34 in Marder). Above that threshold, $\sigma \propto (n - n_c)^\alpha$ with a certain critical exponent.[24] This is an example of insulator to metal transition. In late 1970s, a scaling argument appeared [25] that while there is such a transition in 3D, ideal 2D and 1D systems should be always insulating at arbitrarily weak disorder.

6.3 Berry phase

Berry phase is an (indirectly) measurable quantity related to a certain property of wavefunctions which vary across some parameter space. For cyclotron motion described by Eq. (18), this parameter space is the $\vec{k}$-space and the property is

$$\vec{\Omega} = i\langle u_k|\nabla_k|u_k\rangle, \qquad \gamma_B = \oint \vec{\Omega} \cdot d\vec{k} \tag{38}$$

where $|u_{\vec{k}}\rangle$ are the wavefunctions. This relation is derived in Supplement I of Ref. [26] assuming adiabatic evolution in the parameter space. The Berry phase γ_B picked up by an electron moving on a closed path (cyclotron orbit) in the $\vec{k}$-space can be rewritten as a surface integral of $\vec{F} = \nabla \times \vec{\Omega}$ across the area enclosed by the cyclotron orbit. This property (Berry curvature) is arguably a more generic choice than $\vec{\Omega}$ because it is gauge-independent (it does not change upon transformation $|u_k\rangle \to e^{i\phi(k)}|u_k\rangle$ where $\phi(k)$ is an arbitrary phase which is uniquely defined in the $\vec{k}$-space). Nevertheless, whatever choice we make, γ_B is also gauge-independent.

The way to (indirectly) measure γ_B is afforded for example by Shubnikov-de Haas oscillations. Eq. (19) which gives the positions of maxima of conductivity, contains [27]

$$\gamma = \frac{1}{2} - \frac{1}{2\pi}\gamma_B. \tag{39}$$

Berry curvature $\vec{F}$ is zero for free electrons in vacuum, hence $\gamma_B = 0$, and Eqs. (19,39) then reproduce the energies of Landau levels $E = \hbar\omega_c(n + \frac{1}{2})$. On the other hand, multi-band systems such as graphene (the two important bands are the electron and the hole cones) may in general have non-zero Berry curvature. The Landau levels in graphene, $E = E_0\sqrt{n}$ with integer n and $E_0 = \sqrt{2e\hbar v_F^2 B}$ bear witness of $\gamma_B = \pi$.

Aside from cyclotron motion, the effect of nonzero Berry curvature can (and should, in general, be) included in the semiclassical equations of motion (16). Since the modification reads $\hbar\vec{v} = \nabla_k E(\vec{k}) + \hbar\vec{F} \times d\vec{k}/dt$, the Berry curvature is sometimes said to act as a "magnetic field" in the $\vec{k}$-space as it resembles the Lorentz force in the direct space.

6.4 Quantum dot in magnetic field

Energy levels in a parabolic 2D quantum dot defined by the confinement potential $\frac{1}{2}m\omega_0^2(x^2 + y^2)$ are $E_{n_x,n_y} = \hbar\omega_0(n_x + n_y + 1)$. Clearly, the n-th level is n-fold degenerate (e.g. $E = 3\hbar\omega_0$ can be

obtained by three combinations of n_x, n_y). This degeneracy is lifted by applying magnetic field

$$E_{n,m} = -\frac{1}{2}m\hbar\omega_c + \frac{1}{2}(2n + |m| - 1)\hbar\sqrt{\omega_c^2 + 4\omega_0^2} \tag{40}$$

These energy levels constitute the so called Fock-Darwin spectrum. In the free-2DEG limit ($\omega_0 = 0$), $n = 1, 2, 3, \ldots$ counts the Landau levels and the integer m resolves their degeneracy.

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